Math 21D
Vogler

Arc Length, Unit Tangent Vector
Unit Normal Vector, & Curvature

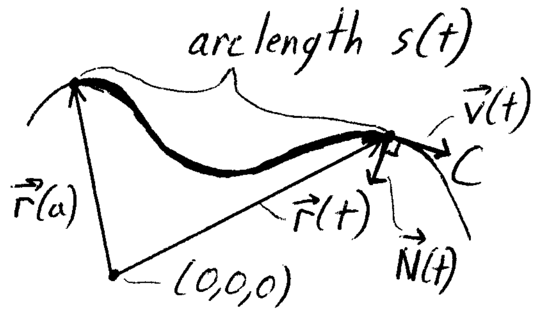
Position Vector: $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Velocity Vector: $\vec{v}(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$

Acceleration Vector: $\vec{a}(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k}$

$$|\vec{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

Let curve C be determined by vector function $\vec{r}(t): \mathbb{R} \to \mathbb{R}^3$.



<u>Defn</u> The <u>arc length</u> s for $t = a$ to t is

$$s(t) = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2} = \int_a^t |\vec{v}(\tau)|\, d\tau$$

<u>Defn</u> The <u>Unit tangent vector</u> for $\vec{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Notes: 1) $\vec{T}(t)$ points in direction of motion along C.
2) $\vec{T}(t)$ is a unit vector.

<u>Defn</u> The <u>principal unit normal vector</u> for $\vec{r}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

<u>Thm</u> 1) $\vec{N}(t)$ is a unit vector
2) $\vec{N}(t)$ is normal to the path C determined by $\vec{r}(t)$,
i.e. $\vec{N}(t)$ is orthogonal to $\vec{T}(t)$.
3) $\vec{N}(t)$ points in the direction that curve C is turning.

Defn The curvature of the path C is

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

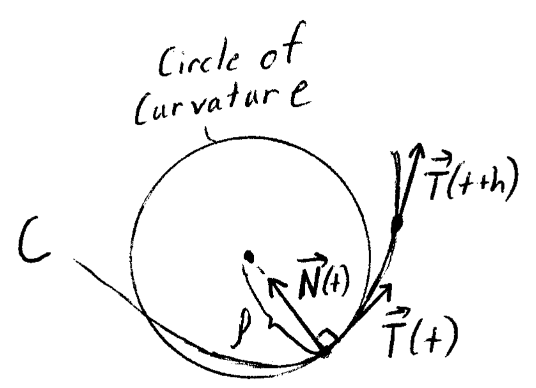
Formula for Computing Curvature

$$\kappa = \frac{1}{|\vec{v}(t)|} \cdot |\vec{T}\,'(t)|$$

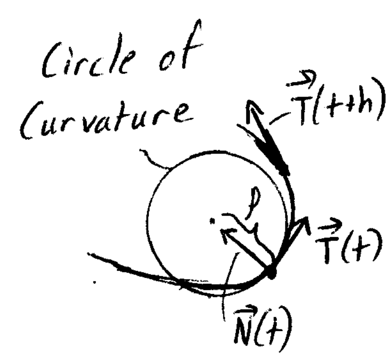
Fact: The curvature of a circle of radius a is $\kappa = \frac{1}{a}$

Defn The circle of curvature at a point P on path C is the circle in the plane of the curve that

1) is tangent to the curve at P (has same tangent line)
2) has same curvature that curve C has at P.
3) lies toward the concave (inner) side of curve C.
4) has radius $\rho = \frac{1}{\kappa}$, which is refered to as radius of curvature



Small κ $\Leftrightarrow$ Big radius ρ
(Since small change in $\vec{T}$)



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