Math 21D

Vogler

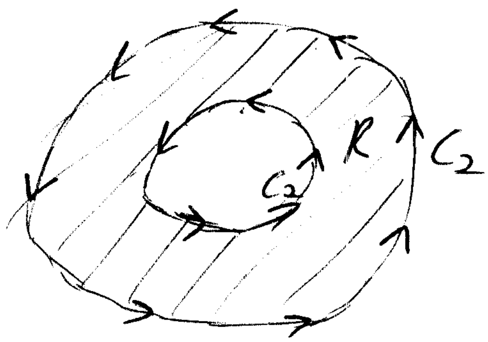
Green's Theorem (Two Curves), and Finding Area using a Line Integral

Green's Theorem 3 (2 Curves Flux - Divergence Normal Form)

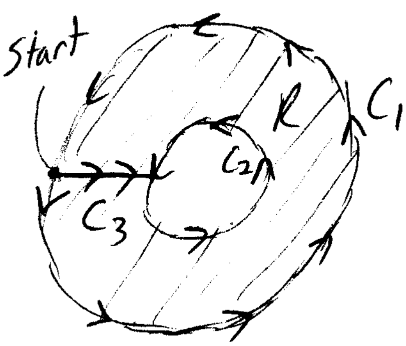
Let R be region in the plane bounded by curves C_1 & C_2, where both C_1 & C_2 are mapped counter-clockwise.

If $\vec{F} = M\vec{i} + N\vec{j}$ is a vector field defined on R

$$\Rightarrow \int_{C_1} \vec{F}\cdot\vec{n}\,ds - \int_{C_2} \vec{F}\cdot\vec{n}\,ds = \iint_R (M_x + N_y)\,dA$$



Proof



Consider path C:

$C_1 \to C_3 \to -C_2 \to -C_3$

By Green's Thm 1, $\iint_R (M_x + N_y)\,dA = \oint_C \vec{F}\cdot\vec{n}\,ds$

$$= \int_{C_1} \vec{F}\cdot\vec{n}\,ds + \int_{C_3} \vec{F}\cdot\vec{n}\,ds + \int_{-C_2} \vec{F}\cdot\vec{n}\,ds + \int_{-C_3} \vec{F}\cdot\vec{n}\,ds$$

$$= \int_{C_1} \vec{F}\cdot\vec{n}\,ds + \int_{C_3} \vec{F}\cdot\vec{n}\,ds - \int_{C_2} \vec{F}\cdot\vec{n}\,ds - \int_{C_3} \vec{F}\cdot\vec{n}\,ds$$

$$= \int_{C_1} \vec{F}\cdot\vec{n}\,ds - \int_{C_2} \vec{F}\cdot\vec{n}\,ds$$

Finding Area Using a Line Integral

Let R be region in plane bounded by curve C

$\Rightarrow$ Area of $R = \frac{1}{2}\oint_C x\,dy - y\,dx$

Proof

Recall, Area of $R = \iint_R 1\,dA$

Suppose $\iint_R (M_x + N_y)\,dA = \iint_R 1\,dA$

$\Rightarrow$ $M_x = \frac{1}{2}$ & $N_y = \frac{1}{2}$ $\Rightarrow$ $M = \frac{1}{2}x$ & $N = \frac{1}{2}y$

$\Rightarrow$ $\vec{F} = \frac{1}{2}x\vec{i} + \frac{1}{2}y\vec{j}$

By Green's Thm 1

$$\iint_R (M_x + N_y)\,dA = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$$

$\Rightarrow$ Area of $R = \iint_R 1\,dA = \iint_R \frac{1}{2} + \frac{1}{2}\,dA = \oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx$

$\Rightarrow$ Area of $R = \frac{1}{2}\oint_C x\,dy - y\,dx$