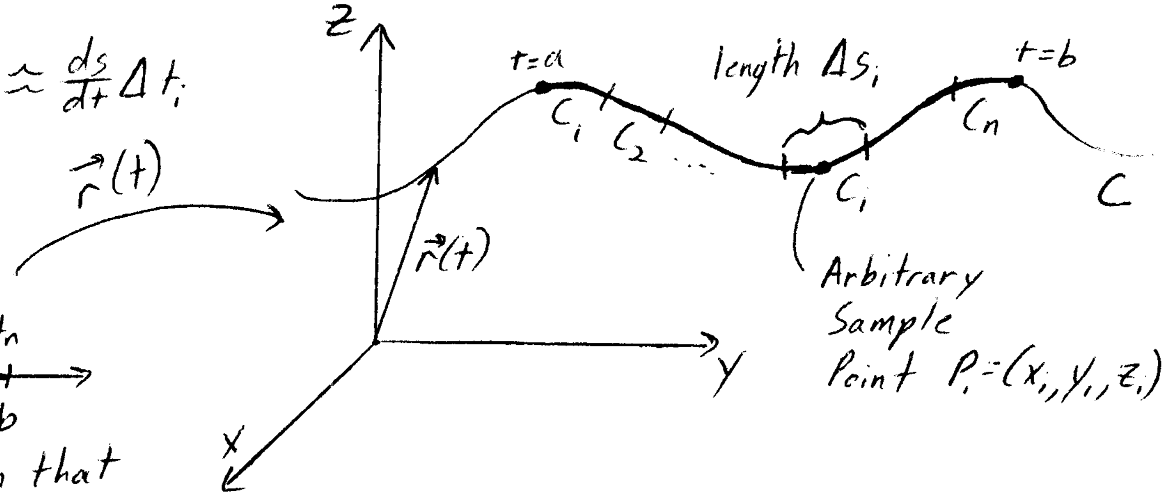
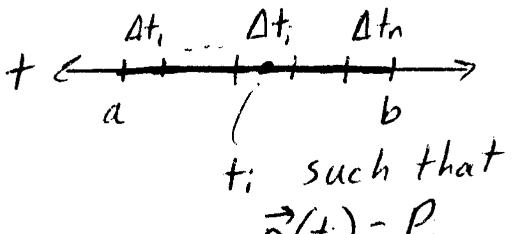
Math 21D
Vogler

Line Integrals

$\vec{r}(t): \mathbb{R} \to \mathbb{R}^3$

Exchange rate: $\Delta s_i \approx \frac{ds}{dt} \Delta t_i$





Defn

- Let C be curve defined by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ for $t=a$ to $t=b$
- Let $f(P)$ be function defined on C (i.e. $f(x,y,z) = f(g(t), h(t), k(t))$)
- Let $C_1, \dots, C_n$ be subdivision of C w/ corresponding arc lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.
- Let $P_i = (x_i, y_i, z_i)$ be arbitrary pt. in C_i $\forall i = 1, \dots, n$
- Let mesh be $\Delta s := \max_{1 \le i \le n} \{\Delta s_i\}$

Then the <u>line integral of f over curve C from $t=a$ to $t=b$ is</u>

$$\int_C f(x,y,z)\,ds := \lim_{\Delta s \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot \Delta s_i$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(g(t_i), h(t_i), k(t_i)) \frac{ds}{dt} \Delta t_i$$

Extra info.

Method of Evaluation

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t)) \cdot \frac{ds}{dt}\,dt$$

w/ $\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(g'(t))^2 + (h'(t))^2 + (k'(t))^2}$

Applications of Line Integrals

Let $\delta(P) = \delta(x,y,z)$ be density ($\frac{mass}{length}$ units) be defined on C.

1) Length $= s := \int_C 1\, ds$

2) Mass $= m := \int_C \delta(P)\, ds$

3) <u>1st Moments about Arbitrary Planes</u>

- About plane $x=\bar{x}$: $M_{x=\bar{x}} = \int_C (x-\bar{x})\,\delta(P)\, ds$
- About plane $y=\bar{y}$: $M_{y=\bar{y}} = \int_C (y-\bar{y})\,\delta(P)\, ds$
- About plane $z=\bar{z}$: $M_{z=\bar{z}} = \int_C (z-\bar{z})\,\delta(P)\, ds$

4) <u>1st Moments about Coordinate Planes</u>

$M_{yz} = \int_C x\,\delta(P)\, ds$, $\quad M_{xz} = \int_C y\,\delta(P)\, ds$, $\quad M_{xy} = \int_C z\,\delta(P)\, ds$

Note: $M_{yz} = M_{x=0}$, $\quad M_{xz} = M_{y=0}$, $\quad M_{xy} = M_{z=0}$

5) <u>Center of Mass $(\bar{x}, \bar{y}, \bar{z})$</u>

$\bar{x} = \dfrac{M_{yz}}{m}$, $\quad \bar{y} = \dfrac{M_{xz}}{m}$, $\quad \bar{z} = \dfrac{M_{xy}}{m}$

6) <u>Centroid $(\bar{x}, \bar{y}, \bar{z})$</u>

$\bar{x} = \dfrac{\int_C x\, ds}{Length}$, $\quad \bar{y} = \dfrac{\int_C y\, ds}{Length}$, $\quad \bar{z} = \dfrac{\int_C z\, ds}{Length}$

7) <u>Moments of Inertia (2nd Moments)</u>

- About line L or pt. P_0: $I = \int_C r^2\,\delta(P)\, ds$
  w/ $r := r(x,y,z)$ is distance from line L or pt. P_0
- About x-axis: $I_x = \int_C (y^2+z^2)\,\delta(P)\, ds$
- About y-axis: $I_y = \int_C (x^2+z^2)\,\delta(P)\, ds$
- About z-axis: $I_z = \int_C (x^2+y^2)\,\delta(P)\, ds$