Vogler
Math 21D

Moments & Center of Mass

Let $\delta(P) = \delta(x,y)$ be density $\left(\frac{\text{mass}}{\text{area}} \text{ units}\right)$.

Total Moments of region R

About line $x=\bar{x}$: $m_{x=\bar{x}} = \iint_R \delta(P)(x-\bar{x})\,dA$

About line $y=\bar{y}$: $m_{y=\bar{y}} = \iint_R \delta(P)(y-\bar{y})\,dA$

First Moments

$$m_y = \iint_R x\,\delta(P)\,dA \qquad m_x = \iint_R y\,\delta(P)\,dA$$

Mass

$$m = \iint_R \delta(P)\,dA$$

Center of Mass

$$\bar{x} = \frac{m_y}{m} \qquad \bar{y} = \frac{m_x}{m}$$

Area

$$A = \iint_R 1\,dA$$

Centroid

$$\bar{x} = \frac{\iint_R x\,dA}{A} \qquad \bar{y} = \frac{\iint_R y\,dA}{A}$$

Moments of Inertia (2nd Moments)

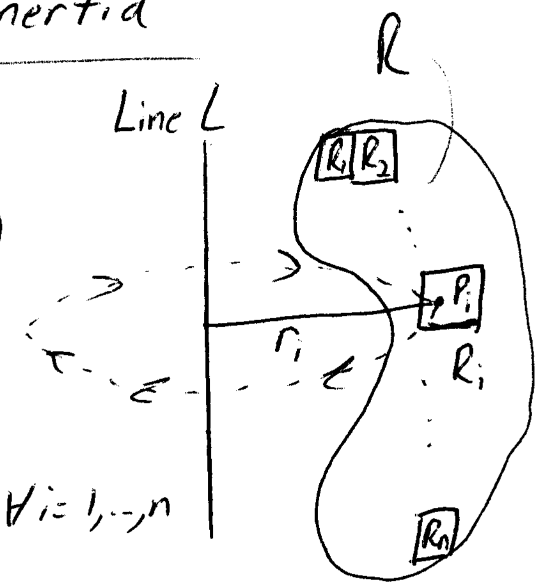
About line L or pt. P_0: $I = \iint_R r^2\,\delta(P)\,dA$

w/ $r = r(x,y)$ is distance from line L or pt. P_0

About x-axis: $I_x = \iint_R y^2\,\delta(P)\,dA$

About y-axis: $I_y = \iint_R x^2\,\delta(P)\,dA$

Finding Rotational Kinetic Energy & Defining Moment of Inertia



- Let R be region of interest.
- Let $\delta(P)$ be density fn on R ($\frac{\text{mass}}{\text{area}}$ units)
- Let $R_1, \ldots, R_n$ be subdivision of R w/ corresponding areas $\Delta A_1, \ldots, \Delta A_n$
- Let $P_i = (x_i, y_i)$ be arbitrary pt. in R_i $\forall i = 1, \ldots, n$
- Let mesh be $\Delta A := \max_{1 \le i \le n} \{\Delta A_i\}$
- Let line L be axis of rotation.
- Let r_i be distance from P_i to line L $\forall i = 1, \ldots, n$

Note: the mass of R_i is approx. $M_i \approx \delta(P_i)\Delta A_i$ $\forall i = 1, \ldots, n$

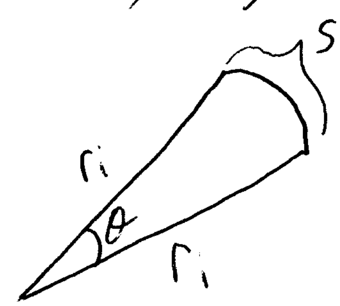
Q: What is the rotational kinetic energy (KE) of the body R?

We start by finding KE of each 'lego' piece R_i.

Recall: $KE = \frac{1}{2} m v^2 = \frac{1}{2}(\text{mass})(\text{speed})^2$

For rotational speed, recall if s is length of an arc on a circle w/ radius r_i, subtended by angle θ, we have

$$s = r_i \theta \overset{\frac{d}{dt}}{\Longrightarrow} \underbrace{\frac{ds}{dt}}_{\text{Linear Speed}} = r_i \underbrace{\frac{d\theta}{dt}}_{\text{Rotational Speed}}$$



Define $\omega := \frac{d\theta}{dt}$ as rotational speed ($\frac{\text{radians}}{\text{sec}}$ units)

$\Rightarrow$ Speed of $R_i = r_i \omega$

Therefore, KE of $R_i = \frac{1}{2} \overbrace{\delta(P_i) \Delta A_i}^{\text{mass}} \overbrace{(r_i \omega)^2}^{\text{speed}^2}$

& Total KE $\approx \sum_{i=1}^{n} \frac{1}{2} \delta(P_i) \Delta A_i \, r_i^2 \omega^2$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} r_i^2 \delta(P_i) \Delta A_i \right) \omega^2$$

By letting mesh go to 0 (i.e $\Delta A \to 0$), we have

$$\text{Total KE} = \frac{1}{2} \iint_R r^2 \delta(P) \, dA \; \omega^2 = \frac{1}{2} I \omega^2$$

where we define moment of inertia as

$$I := \iint_R r^2 \delta(P) \, dA$$

Note that: Linear KE $= \frac{1}{2} m v^2$

& Rotational KE $= \frac{1}{2} I \omega^2$

Conclusion: The moment of inertia acts like rotational mass in terms of kinetic energy. It is a measure of an object's resistance to changes in its rotation.