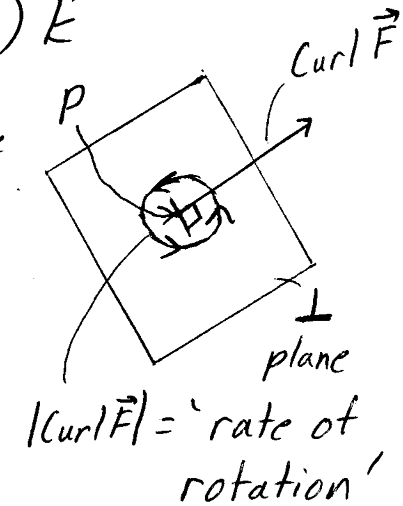
Curl & Stoke's Theorem

Defn Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field. The <u>curl of $\vec{F}$</u> is the vector field

$$\operatorname{curl}\vec{F} := \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$\Rightarrow \operatorname{curl}\vec{F} = (P_y - N_z)\vec{i} - (P_x - M_z)\vec{j} + (N_x - M_y)\vec{k}$$

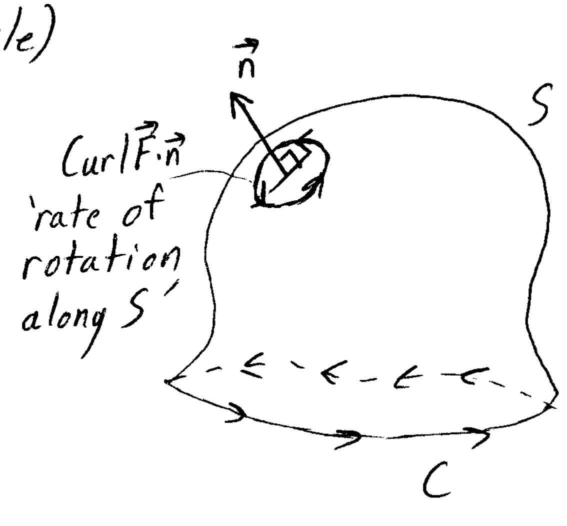
Remark: It can be shown the magnitude of the curl of $\vec{F}$ at point $P=(x,y,z)$ is a measure of the rate of rotation of the fluid relative to the plane orthogonal to Curl $\vec{F}$ (See Figure).



Stoke's Theorem

Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field defined on surface S with boundary C oriented counter-clockwise relative to the surface's unit normal vector $\vec{n}$ (right-hand rule)

$$\Rightarrow \underbrace{\oint_C \vec{F}\cdot\vec{T}\,ds}_{\substack{\text{Counterclockwise} \\ \text{Circulation}}} = \underbrace{\iint_S \underbrace{\vec{\nabla}\times\vec{F}}_{\text{Curl}\vec{F}}\cdot\vec{n}\,d\sigma}_{\substack{\text{Curl Integral} \\ \text{or} \\ \text{Total Interior} \\ \text{C.C. Rotations}}}$$



Notice that when $\text{Curl}\,\vec{F} = 0$

$\Leftrightarrow \quad P_y = N_z, \quad M_z = P_x, \quad N_x = M_y$

$\Leftrightarrow \quad \vec{F}$ is conservative

which leads to following result

Theorem

If $\text{Curl}\,\vec{F} = 0$ on region $D \Rightarrow$ for any closed curve C in D

$$\oint_C \vec{F} \cdot \vec{T}\, ds = 0$$

Combined with previous theorems (Refer to 'Conservative Vector Fields' handout), we now have the following equivalencies:

$\vec{F}$ is conservative on D $\Leftrightarrow$ $\vec{F} = \vec{\nabla} f$ on D (There exists potential function of $\vec{F}$)

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ $\forall$ closed curves C in D $\Leftrightarrow$ $\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = 0$ throughout D

Useful Identity For any function $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$$\text{Curl}(\text{Grad}\, f) = 0 \quad \text{or} \quad \vec{\nabla} \times \vec{\nabla} f = 0$$