Math 21D
Vogler

Surface Integrals

Defn

- Let S be surface parametrized by $\vec{r}(u,v): \mathbb{R}^2 \to \mathbb{R}^3$, (Refer to previous handout) where $\vec{r}$ maps R to S.
- Let $f(P)$ be function defined on S
- Let $S_1, S_2, \dots, S_n$ be subdivion of S w/ corresponding surface areas $\Delta\sigma_1, \Delta\sigma_2, \dots, \Delta\sigma_n$.
- Let $P_i = (x_i, y_i, z_i)$ be arbitrary pt. in S_i $\forall i = 1, \dots, n$
- Let mesh be $\Delta\sigma := \max_{1 \le i \le n} \{\Delta\sigma_i\}$

Then the surface integral of f over surface S is

$$\iint_S f(x,y,z)\,d\sigma := \lim_{\Delta\sigma \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta\sigma_i$$

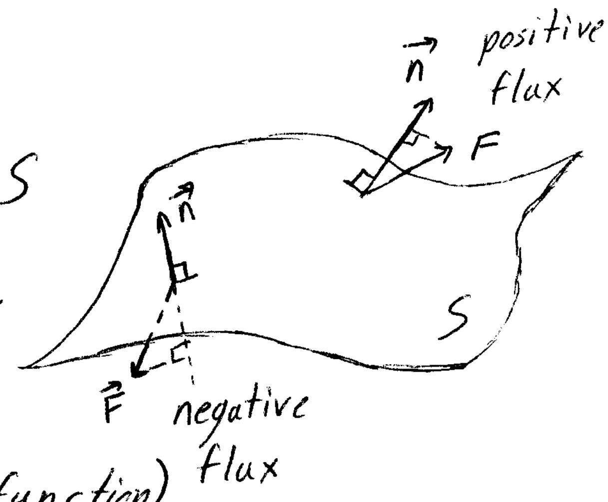
Method of Evaluation

$$\iint_S f(x,y,z)\,d\sigma = \iint_R f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,dA = \iint_R f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

Note: For implicit functions, you must use the 'implicit fuction exchange rate' instead of $|\vec{r}_u \times \vec{r}_v|$ (Refer to previous handout)

Defn The flux of vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ across surface S in the direction $\vec{n}$ (This vector is is a chosen unit normal vector) is

$$Flux = \iint_S \vec{F} \cdot \vec{n}\,dS$$



Note: If $S: F(x,y,z) = C$ (ie. implicit function)

$$\Rightarrow \quad \vec{n} = \frac{\vec{\nabla} F}{|\vec{\nabla} F|}$$ (this F is different from $\vec{F}$ a v-field)

Applications of Surface Integrals

Let $\delta(P) = \delta(x,y,z)$ be density ($\frac{mass}{area}$ units) defined on S

1) Surface Area $= SA := \iint_S 1\, d\sigma$

2) Mass $= M := \iint_S 1\, d\sigma$

3) **1st Moments about Arbitrary Planes**

- About plane $x=\bar{x}$: $M_{x=\bar{x}} = \iint_S (x-\bar{x})\,\delta(P)\,d\sigma$
- About plane $y=\bar{y}$: $M_{y=\bar{y}} = \iint_S (y-\bar{y})\,\delta(P)\,d\sigma$
- About plane $z=\bar{z}$: $M_{z=\bar{z}} = \iint_S (z-\bar{z})\,\delta(P)\,d\sigma$

4) **1st Moments about Coordinate Planes**

$M_{yz} = \iint_S x\,\delta(P)\,d\sigma$, $\quad M_{xz} = \iint_S y\,\delta(P)\,d\sigma$, $\quad M_{xy} = \iint_S z\,\delta(P)\,d\sigma$

Note: $M_{yz} = M_{x=0}$, $\quad M_{xz} = M_{y=0}$, $\quad M_{xy} = M_{z=0}$

5) **Center of Mass $(\bar{x},\bar{y},\bar{z})$**

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

6) **Centroid $(\bar{x},\bar{y},\bar{z})$**

$$\bar{x} = \frac{\iint_S x\,d\sigma}{Area}, \quad \bar{y} = \frac{\iint_S y\,d\sigma}{Area}, \quad \bar{z} = \frac{\iint_S z\,d\sigma}{Area}$$

7) **Moments of Inertia (2nd Moments)**

- About line L or pt. P_0: $I = \iint_S r^2\,\delta(P)\,d\sigma$
  w/ $r := r(x,y,z)$ is distance from line L or pt. P_0
- About x-axis: $I_x = \iint_S (y^2+z^2)\,\delta(P)\,d\sigma$
- About y-axis: $I_y = \iint_S (x^2+z^2)\,\delta(P)\,d\sigma$
- About z-axis: $I_z = \iint_S (x^2+y^2)\,\delta(P)\,d\sigma$