Math 21D
Vogler

Vector Field, Gradient Field, Line Integral along $\vec{F}$, Work, & Flow

Defn A vector field is a function which assigns a vector to each point in space (aliases include force field, velocity field, gradient field, gravitational field, ...)

In 2D) $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2 \quad \vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$

In 3D) $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3 \quad \vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$

Defn The gradient field of a scalar function $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$ is vector field

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

Recall: Let $z = f(x,y)$ be surface in 3D $\Rightarrow$ gradient vector $\vec{\nabla} f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$ has the following properties:

1) $\perp$ to the level curve $f(x,y) = C$
2) points in the direction of maximum directional derivative.
3) has magnitude equal to value of maximum directional derivative.

Defn Let $\vec{F}$ be vector field defined on curve C, where C is defined by $\vec{r}(t)$ for $a \le t \le b$. Then the line integral of $\vec{F}$ along $\vec{r}(t)$ (or C) is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F} \cdot d\vec{r}$$

Method of Evaluation

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t))\, \vec{r}\,'(t)\, dt$$

Different Notations for Line Integral

Let vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be defined on curve $C: \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ $a \le t \le b$

$\Rightarrow \int_C \vec{F}\cdot\vec{T}\,ds$ The definition

$= \int_C \vec{F}\cdot d\vec{r}$ Vector differential form

$= \int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$ Parametric vector evaluation

$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt$ Parametric scalar evaluation

$= \int_C M\,dx + N\,dy + P\,dz$ Scalar differential form

Note: The evaluation's are more useful for computation, while the form's are more useful in theorectical proofs.

Applications

1) If $\vec{F}$ is a field of force vectors (aka force field), $\vec{F}$ on $\vec{r}(t)$ is
$\Rightarrow$ the work done by

$$\text{Work} = W := \int_C \underbrace{\vec{F}\cdot\vec{T}}_{\text{Force in direction of curve } C}\ \underbrace{ds}_{\text{distance}}$$

2) If $\vec{F}$ is a velocity field (for fluid flows) for $C: \vec{r}(t)$ for $a \le t \le b$ is
$\Rightarrow$ the flow along curve

$$\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds$$

(If C is closed $(\vec{r}(a) = \vec{r}(b))$ this flow is called circulation around the curve C)