MAT 17A - UHP - DISCUSSION #10
December 1, 2015

PROBLEM 1. Linear recursion equations

Consider the linear recursion (finite difference) equation

$$x_{j+1} = \mathcal{L}(x_j) = a\, x_j + b, \quad x_0 \qquad \dots (1),$$

where a and b are:

(i) $a = 0.5,\ b = 1.$ (ii) $a = 1.1,\ b = -0.2.$ (iii) $a = -0.6,\ b = 3.2.$

[a] Find the fixed point for each parameter set (i - iii) and determine their stabilities based on the value of a.

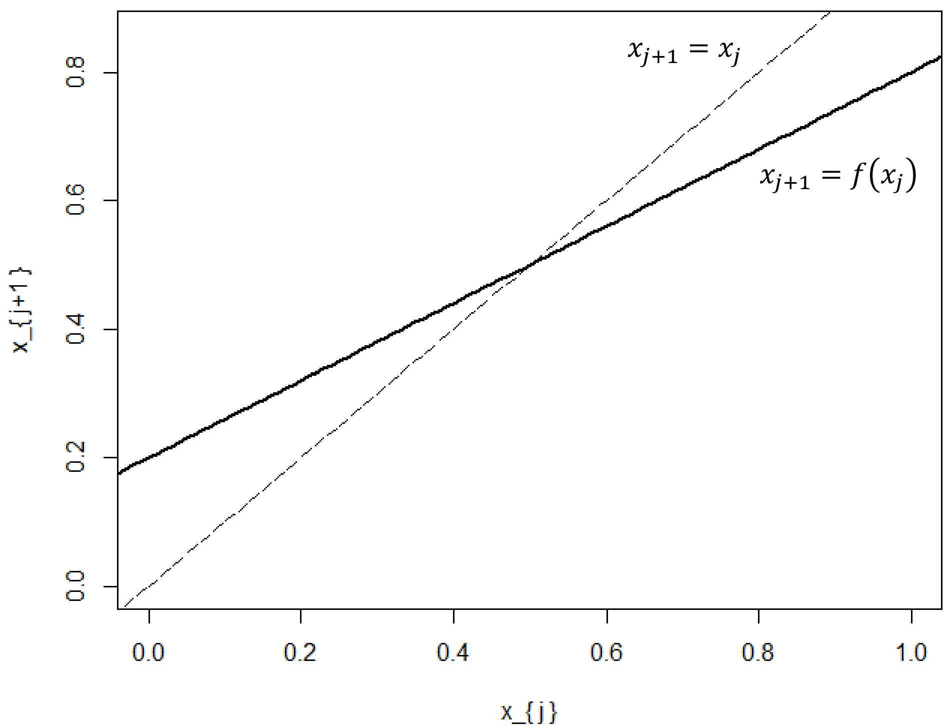
[b] Use R/Rstudio to numerically iterate the recursion equation for each parameter set and verify your answers in [a]. (Choose initial conditions x_0 close to and far from the fixed point and see what happens.)

R-code to iterate recursion equation (1):

Click on icon with green plus sign in upper left-hand of RStudio window just below "File". Choose R-script, and a new panel (the source code panel) will open up in RStudio. Type (or copy and paste) the entire code below into the source code panel. To run the program (i.e., execute all command in the code), click on "Source" in the top right-hand corner of the source code panel. Edit parameters and initial condition as needed.

```
a=0.5                                     # set the value of parameter a
b=1.0                                     # set the value of parameter b
x = 0                                     # initialize sequence x[j]
x[1] = 0.1                                # set the initial condition
for (j in 1:20) {                         # set number of computed sequence terms
  x[j+1] = a * x[j] + b  }                # define the recurrence relationship
plot(x,type='b',xlab='j',ylab='x_{j}')    # plot output
```

PROBLEM 2. Graphical iteration of recursion equations



The thick solid line on the graph above plots $x_{j+1} = f(x_j)$. The thin dashed line on the graph plots the "identity line", $x_{j+1} = x_j$.

Use **the graph itself** (not the equation for any of the curves) to do the following:

(i) Determine the fixed point of $x_{j+1} = f(x_j)$.

(ii) Determine the stability of the fixed point. (Note the slope of $f(x_j)$ at the fixed point. Is it greater than 1 in absolute value or less than 1 in absolute value?)

(iii) Given $x_0 = 0$, approximate x_1, x_2, x_3, and x_4.

(iv) Given $x_0 = 1.0$, approximate x_1, x_2, x_3, and x_4.

PROBLEM 3. The discrete logistic equation - an example of a nonlinear recursion equation

Consider the linear recursion (finite difference) equation

$$x_{j+1} = f(x_j) = r\,(1 - x_j)\,x_j\,, \quad x_0 \qquad \ldots (2),$$

where r is a constant between 0 and 4, and $0 \leq x_0 \leq 1$.

[a] Show that the logistic equation has the fixed points $x^* = 0$ and $x^* = 1 - 1/r$. Can you easily determine the stability of the fixed points?

[b] Use R/Rstudio to numerically iterate and explore the effects of changing the value of r.
Use $r = 0.5,\ r = 0.9,\ r = 1.1,\ r = 1.5,\ r = 2.0,\ r = 2.5,\ r = 2.8\,,\ r = 3.2,\ r = 3.52,\ r = 3.56,$ $r = 3.567,\ r = 3.569,\ r = 3.8\,,\ r = 3.9\,.$ When are the fixed points stable and when are they unstable? Describe any interesting behavior that you find.

R-code to iterate recursion equation (2):

Click on icon with green plus sign in upper left-hand of RStudio window just below "File". Choose R-script, and a new panel (the source code panel) will open up in RStudio. Type (or copy and paste) the entire code below into the source code panel. To run the program (i.e., execute all command in the code), click on "Source" in the top right-hand corner of the source code panel. Edit parameters and initial condition as needed.

```
r=0.5                                  # set the value of parameter r
x = 0                                  # initialize sequence x[j]
x[1] = 0.1                             # set the initial condition
for (j in 1:29) {                      # set number of computed sequence terms
  x[j+1] = r * (1-x[j]) * x[j]   }     # define the recurrence relationship
plot(x,type='b',xlab='j',ylab='x_{j}') # plot output
```

PROBLEM 4. The discrete logistic equation - linear stability analysis

Consider the logistic equation with $r = 3/2 = 1.5$,

$$x_{j+1} = f(x_j) = \frac{3}{2}\,(1 - x_j)\,x_j\,,$$

with $0 \leq x_0 \leq 1$.

[a] Show that the logistic equation has the fixed points $x^* = 0$ and $x^* = 1/3$.

[b] Determine the linear approximation of $f(x_j)$ around $x^* = 0$. Use the this linearization to approximate the recursion equation near $x^* = 0$. Based on this approximation, what is the stability of the fixed point $x^* = 0$?

[c] Determine the linear approximation of $f(x_j)$ around $x^* = 1/3$. Use the this linearization to approximate the recursion equation near $x^* = 1/3$. Based on this approximation, what is the stability of the fixed point $x^* = 1/3$?

Do your answers in [b] and [c] agree with your simulations in 2[b] for $r = 1.5$.

HOMEWORK PROBLEM 1. The discrete logistic equation - linear stability analysis

Consider the logistic equation for general r,

$$x_{j+1} = f(x_j) = r\,(1 - x_j)\,x_j\ ,$$

with $0 \leq x_0 \leq 1$.

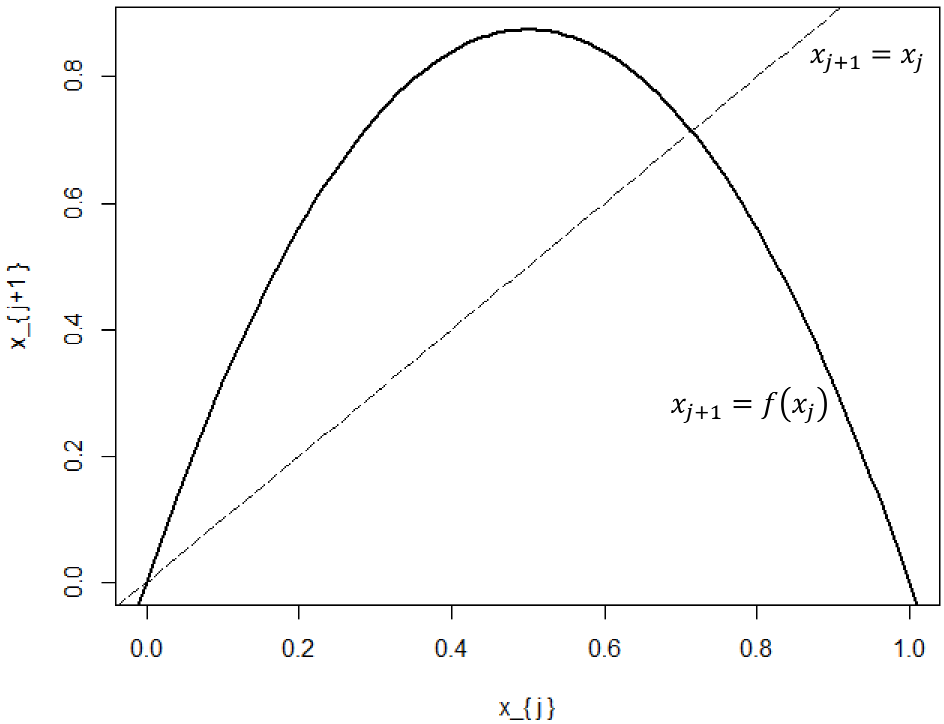
Discussion sheet 10 problem 3 [a], you showed that the logistic equation has the fixed points $x^* = 0$ and $x^* = 1 - 1/r$.

[b] Determine the linear approximation of $f(x_j)$ around $x^* = 0$. Use the this linearization to approximate the recursion equation near $x^* = 0$. Based on this approximation, what is the stability of the fixed point $x^* = 0$?

[c] Determine the linear approximation of $f(x_j)$ around $x^* = 1 - 1/r$. Use the this linearization to approximate the recursion equation near $x^* = 1 - 1/r$. Based on this approximation, what is the stability of the fixed point $x^* = 1 - 1/r$? [Note: $-1 < 2 - r < 1$ is equivalent to $1 < r < 3$.]

Do your answers in [b] and [c] agree with your simulations in Discussion sheet 10 problem 2[b].

HOMEWORK PROBLEM 2. Graphical iteration of a recursion equation



The thick solid line on the graph above plots $x_{j+1} = f(x_j)$. The thin dashed line on the graph plots the "identity line", $x_{j+1} = x_j$.

Use **the graph itself** (not the equation for any of the curves) to do the following:

(i) Determine the fixed points of $x_{j+1} = f(x_j)$, and determine the stability of the fixed points.

(ii) Given $x_0 = 0.1$, approximate $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$, and x_{10}.

HOMEWORK PROBLEM 3. Dynamics of Medication in bloodstream

[a] A patient is a dose of 1 mg of a strong pain medication intravenously every hour. 25% of the drug is absorbed by the tissue, broken down or expelled from the bloodstream every hour.

(i) Write down the recursion relation with the initial condition that describes the amount of drug in the patient's bloodstream over time (i.e., find the equation in the form $x_{k+1} = f(x_k),\ x_0$). Be sure to define your variables and their units.

(ii) Find the fixed point(s) of the system and determine if the fixed point(s) is(are) stable or unstable.

(iii) Interpret your findings in (ii) biologically.

[b] Often the amount of drug absorbed by the tissue, broken down or expelled from the bloodstream depends on the concentration of drug in the bloodstream. Suppose that fraction of drug absorbed by the tissue, broken down or expelled from the bloodstream each hour $\rho(x_k)$ is given by

$$\rho(x_k) = \frac{0.5}{1.0 + 0.1\, x_k}.$$

(i) Interpret $\rho(x_k)$ biologically.

(ii) Write down the recursion relation with the initial condition that describes the amount of drug in the patient's bloodstream over time.

(iii) Find the fixed point(s) of the system and determine if the fixed point(s) is(are) stable or unstable.

(iv) Interpret your findings in (iii) biologically.