MAT 17A - DISCUSSION #9

November 24, 2015

Problem 1. DNA Synthesization

The probability distribution of waiting times (t) for two new strands of DNA to be synthesized by a DNA polynerase is given by

$$f(t) = k^2 t e^{-kt}, \quad t \geq 0,$$

where k is a positive constant.

[For full credit, you must solve the following problems for general k. For partial credit, you can solve these problems with $k = 1$.]

(a) Find

$$\lim_{t \to \infty} f(t) \quad \text{and} \quad \lim_{t \to 0} f(t).$$

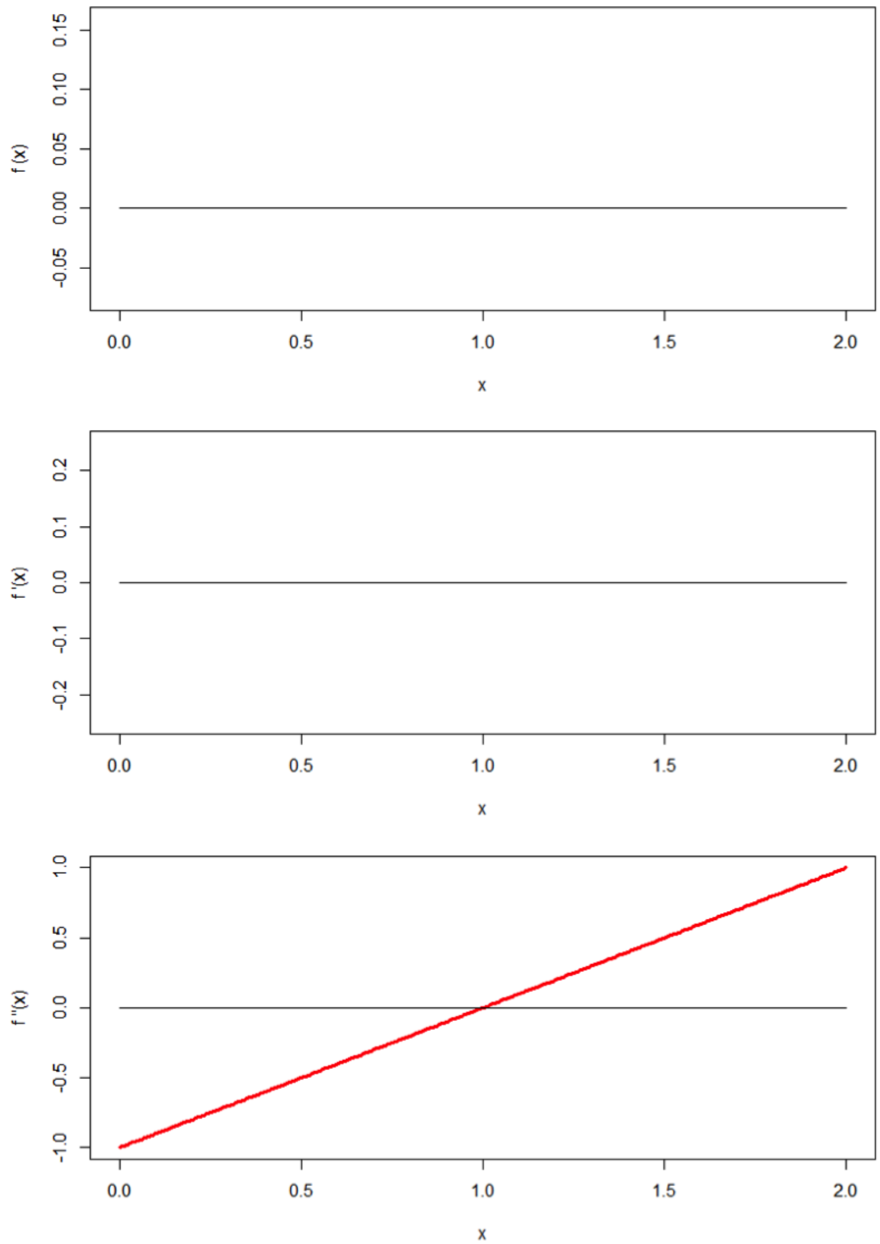
Justify your answers.

(b) Determine the intervals of $t \geq 0$ in which $f(t)$ is increasing and in which it is decreasing.

(c) Find the most frequent waiting time, i.e., find the global maximum of $f(t)$ for $t \geq 0$. (Use $f'(t)$ to argue that your answer does indeed correspond to a *global* maximum.)

Problem 2. Sketching Functions based on their Derivatives

(a) Based on the graph of $f''(x)$, sketch a corresponding graph of $f'(x)$ and $f(x)$.



(b) What does $f''(x)$ say about $f'(x)$ and how does that translate into what it says about $f(x)$?

Problem 3. Dose-Response of Multi-Vitamins

Multi-vitamins typically have dose-response curves of the following form

$$R = f(x) = \frac{ax}{k^2 + x^2}, \quad x \geq 0,$$

where x is a measure of the daily dose, R is a measure of the health benefits, and a and k are positive constants.

(a) Find $f(0)$ and determine the sign of $f(x)$ for $x > 0$.

(b) Find $\lim_{x\to\infty} f(x)$ and determine whether $f(x)$ has a horizontal asymptote.

(c) Determine where $f(x)$ is increasing and where it is decreasing.

(d) Determine where $f(x)$ is concave up and where it is concave down.

(e) Find all local minima and maxima and inflection points.

(f) Graph $f(x)$ using the information you obtained in (a) - (e).

Note that $f'(x) = \frac{a(k^2-x^2)}{(k^2+x^2)^2}$ and $f''(x) = \frac{a2x(x^2-3k^2)}{(k^2+x^2)^3}$

(g) Interpret your graph in terms of the health benefits of the multi-vitamins.

Problem 4. Homework Problem

Engineers and physicists often use the simplifying assumption that, for small angle θ, $\sin\theta$ is approximately equal to θ.

(a) Find the linear approximation for $\sin\theta$ near $\theta = 0$ that shows why this is a reasonable assumption.

(b) Using RStudio, plot $y = \sin\theta$ and $y = \theta$ on the same graph to compare the approximation with the actual function.

(c) At what values of θ is the linear approximation accurate to 5%?

(d) Using the laws of classical mechanics, the motion of a "simple" pendulum is described by

$$\frac{d^2\theta}{dt^2} = -k\sin(\theta),$$

where $k = g/L$, g is the gravitational constant $9.8 m/sec^2$, L is the length of the pendulum arm, θ is the angular displacement, and t is time.
(See https://en.wikipedia.org/wiki/Pendulum_(mathematics)). If the angular displacement of the pendulum is small, the motion of the pendulum is given by

$$\frac{d^2\theta}{dt^2} = -k\theta.$$

What is a (non-zero) solution to this "differential equation"? Does your answer make physical sense?

Problem 5. Homework Problem

(a) You are studying a particular population of bacteria in your lab. You notice that the bacteria synchronously divide into two daughter cells every 20 minutes, but only a fraction ρ of these daughter cells survive due to the presence of an antibiotic. The bacterial population started off with N_0 bacterial cells immediately before the first division. Construct a recursion equation of the form $N_{k+1} = f(n_k)$ that models this situation.

(b) Suppose that there was a net influx of bacteria into the colony at a rate of b bacteria/20 minutes. Assume that all bacterial fatalities occur during division. Modify your recursion equation from (a) to take this influx into account.

Problem 6. Homework Problem

[I] Consider the following recursion equation

$$N_{k+1} = rN_k.$$

(a) Suppose that the initial condition is $N_0 = 0$ What happens to N_k as k increases? Does this behavior depend on the parameter r?

(b) Suppose $N_0 \neq 0$. What happens to N_k as k increases? Describe how the behavior depends on the parameter r? (Find the solution to the recursion equation to justify your answer.)

[II] Consider the following recursion equation

$$N_{k+1} = rN_k + b.$$

(a) What happens to N_k as k increases when the initial condition is $N_0 = b/(1-r)$? Does this behavior depend on the parameter r?

(b) Suppose $N_0 \neq b/(1-r)$. What happens to N_k as k increases? Describe how the behavior depends on the parameter r?

To answer (b), define $N_k = x_k + b/(1-r)$ and substitute this into the recursion equation to obtain $x_{k+1} = rx_k$. Note that this is the same equation examined in [I]. Solve this recursion equation for x_k, and then obtain N_k.