*Instr: Zeke Vogler* *October 21, 2015*

*Name:* Key *Exam ID:*__________

**PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!**

1. Make sure that your exam contains 7 pages, including this one.
2. **NO** calculators, books, notes, other written material, or help from other students allowed.
3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
4. You may NOT use L'Hopital's Rule to determine limits on this exam.
5. You may NOT use shortcuts from the textbook to determine limits to infinity on this exam.
6. You will be graded on proper use of sequence, limit, and indeterminate form notation.
7. You must put units on answers where units are appropriate.
8. You have until 1:00pm to finish this exam.
9. Read the statement below and sign your name.

   *I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.*

Signature: ______________________________

*"You can profit from your mistakes, but that does not mean the more mistakes, the more profit." – Anonymous*

*GOOD LUCK!!!*

1. (*6 pts*) State the IN CLASS (informal) definition of the limit for

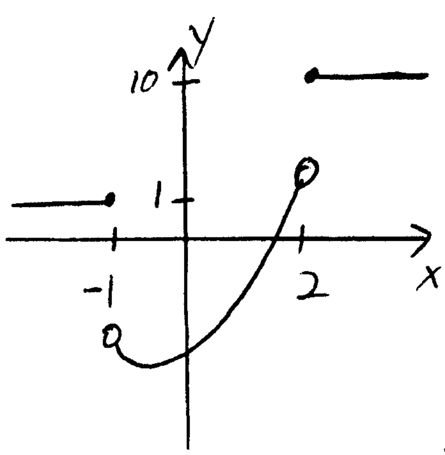
$$\lim_{x \to a} f(x) = L$$

Let $y = f(x)$ be a function. The <u>limit</u> as $x$ approaches a number 'a' equals $L$ means as x-values get closer to 'a', the corresponding y-values get closer to $L$.

2. (*10 pts*) Consider the following function

$$f(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ Ax^2 + Bx & \text{if } -1 < x < 2 \\ 10 & \text{if } x \geq 2 \end{cases}$$

Use limits and a "fake graph" to determine the value of constants $A$ and $B$ so that the function is continuous for all values of $x$ .




We need for continuity ('kiss condition')

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$ & $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$\Rightarrow 1 = \lim_{x \to -1^+} Ax^2 + Bx$ & $\lim_{x \to 2^-} Ax^2 + Bx = 10$

$\Rightarrow A - B = 1$ & $4A + 2B = 10$

$\Rightarrow \underline{A = B + 1}$ & $\underline{2A + B = 5}$

sub

$\Rightarrow 2(B+1) + B = 5 \Rightarrow 3B + 2 = 5 \Rightarrow 3B = 3$

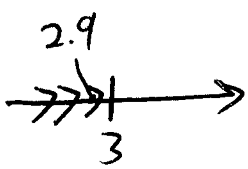
$\Rightarrow \boxed{B = 1 \text{ \& } A = 1 + 1 = 2}$

3. (*6 pts each*) Determine the following limits

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x} = `\dfrac{0}{0}'$

$\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x} \cdot \dfrac{3}{3} = \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x} \cdot \dfrac{3}{4} = 1 \cdot \dfrac{3}{4} = \boxed{\dfrac{3}{4}}$

(b) $\lim\limits_{x\to 3^-} \dfrac{x+2}{x-3} = `\dfrac{5}{0^-}' = \boxed{-\infty}$




(c) $\lim\limits_{x\to\infty} \sqrt{x+100} - \sqrt{x} = `\infty - \infty'$

$\lim\limits_{x\to\infty} \sqrt{x+100} - \sqrt{x} \cdot \dfrac{\sqrt{x+100}+\sqrt{x}}{\sqrt{x+100}+\sqrt{x}} = \lim\limits_{n\to\infty} \dfrac{(\cancel{x}+100)-\cancel{x}}{\sqrt{x+100}+\sqrt{x}}$

$= \lim\limits_{x\to\infty} \dfrac{100}{\sqrt{x+100}+\sqrt{x}} = `\dfrac{100}{\infty}' = 0$

(d) $\lim\limits_{x\to\infty} \dfrac{e^x+1}{2e^x+4} \cdot \dfrac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \dfrac{1+\cancel{\frac{1}{e^x}}^{0}}{2+\cancel{\frac{4}{e^x}}^{0}} = \dfrac{1}{2}$

$`\dfrac{\infty}{\infty}' \;//$

4. (*10 pts*) Use the Squeeze Principle (Sandwich Theorem) to determine the limit of the following sequence:

$$a_n = \frac{n \sin n}{n^2+1}$$

$-1 \le \sin n \le 1$

$\Rightarrow -n \le n \sin n \le n$

$\Rightarrow \frac{-n}{n^2+1} \le \frac{n \sin n}{n^2+1} \le \frac{n}{n^2+1}$

$\frac{\infty}{\infty}{}' = \lim_{n\to\infty} -\frac{n}{n^2+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-\cancel{1/n}^{\,0}}{1+\cancel{1/n^2}^{\,0}} = 0$

$\parallel$

$\frac{\infty}{\infty}{}' = \lim_{n\to\infty} \frac{n}{n^2+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\cancel{1/n}^{\,0}}{1+\cancel{1/n^2}^{\,0}} = 0$

By Squeeze Principle

$\Rightarrow \lim_{x\to\infty} \frac{n \sin n}{n^2+1} = 0$

5. (*8 pts total*) Let $X = \log x$ and $Y = \log y$ for the following problems

(a) (*5 pts*) When the following table data is graphed on a log-log plot (i.e. using $X$ and $Y$ coordinates), a straight line results. Determine the equation of this line and graph the resulting line on a log-log plot.

| $x$ | $X$ | $y$ | $Y$ |
|---|---|---|---|
| 1 | 0 | 10000 | 4 |
| 10 | 1 | 100 | 2 |

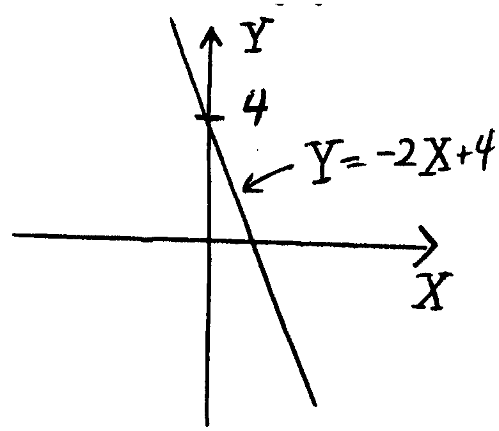
Using points $(0,4)$ & $(1,2)$,

$m = \frac{Y_1 - Y_2}{X_1 - X_2} = \frac{4-2}{0-1} = -2$,

$Y$-int: $b = 4 = \log 10{,}000$

$\Rightarrow Y = mX + b = -2X + 4$

$\Rightarrow Y = -2X + 4$




(b) (*3 pts*) Assume in part (a) your equation of a line was $Y = -4X + 1$, use the appropriate logarithmic transformation to find the function relationship between $x$ and $y$.

$Y = -4X + 1 \Rightarrow \log y = -4 \log x + \log 10 = \log x^{-4} + \log 10$

$\Rightarrow \log y = \log 10x^{-4} \Rightarrow \boxed{y = 10x^{-4}}$

6. (*5 pts each*) Determine the $n$th term (starting with $n = 0$) of each of the following sequences.

(a) $\frac{1}{3}, 0, \frac{-1}{7}, \frac{-2}{9}, \frac{-3}{11}, \cdots$

$-1 \quad -1 \quad -1$

$? \quad ? \quad +2 \quad +2$

$$\boxed{a_n = \frac{1-n}{3+2n}}$$

(b) $-\frac{1}{9}, \frac{1}{3}, -1, 3, -9, 27, \cdots$

$\cdot 3 \quad \cdot 3 \quad \cdot 3$

$$\boxed{a_n = (-1)^{n+1} \frac{1}{9} 3^n}$$

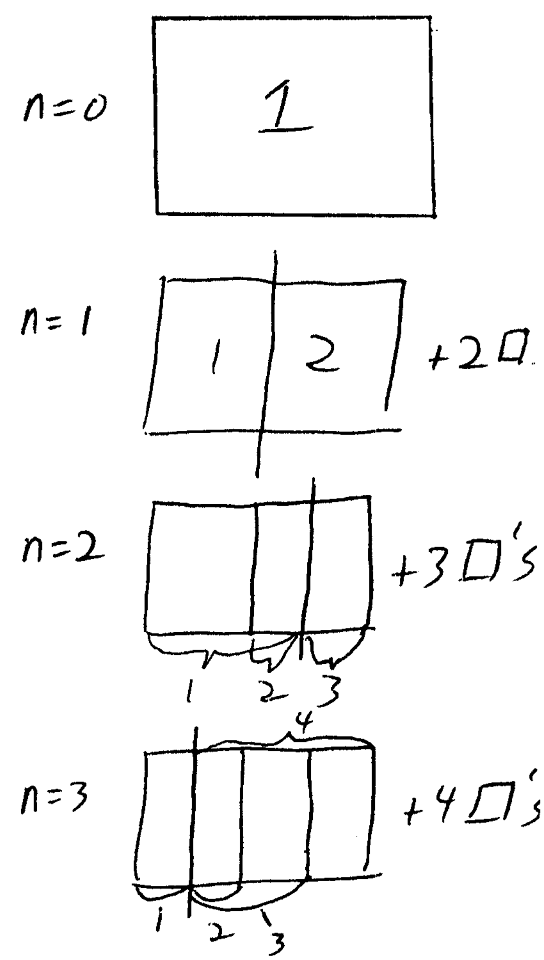
7. (*12 pts*) What is the maximum number of rectangles which can be formed within the boundary of the given figure using 99 vertical lines? Count all rectangles including overlapping ones. (HINT: Use the fact that $1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$.)

| # lines ($n$) | # rects ($a_n$) |
|---|---|
| 0 | 1 |
| 1 | $1+2=3$ |
| 2 | $1+2+3=6$ |
| 3 | $1+2+3+4=10$ |
| $\vdots$ | $\vdots$ |
| $n$ | $1+2+\ldots+n+(n+1)$ |

$$\Rightarrow a_n = \frac{(n+1)(n+2)}{2}$$

For $n = 99$, $a_{99} = \frac{100(101)}{2}$ rectangles

$$= \boxed{5{,}050 \text{ rectangles}}$$



8. (*8 pts total*) Consider the following function $f(x) = \dfrac{x+1}{2-x}$

(a) (*3 pts*) Show algebraically that f is one-to-one.

Assume $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1+1}{2-x_1} = \dfrac{x_2+1}{2-x_2} \Rightarrow (x_1+1)(2-x_2) = (x_2+1)(2-x_1)$

$\Rightarrow 2x_1 + \cancel{2} - \cancel{x_1x_2} - x_2 = 2x_2 + \cancel{2} - \cancel{x_1x_2} - x_1 \Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$. Thus, f is 1-1.

(b) (*5 pts*) Determine $y = f^{-1}(x)$, the inverse function for $y = f(x)$.

$y = \dfrac{x+1}{2-x} \Rightarrow$ (switch variables) $x = \dfrac{y+1}{2-y} \Rightarrow (2-y)x = y+1$

$\Rightarrow 2x - xy = y+1 \Rightarrow -y - xy = 1-2x \Rightarrow -y(1+x) = 1-2x$

$\Rightarrow y = \dfrac{2x-1}{x+1} \Rightarrow \boxed{f^{-1}(x) = \dfrac{2x-1}{x+1}}$

9. (*6 pts*) Determine all possible fixed points for the following recursion : $a_{n+1} = \sqrt{2a_n}$

$a_{n+1} = \sqrt{2a_n} \Rightarrow L = \sqrt{2L} \Rightarrow L^2 = 2L \Rightarrow L^2 - 2L = 0$

$\Rightarrow L(L-2) = 0 \Rightarrow \boxed{L = 0, 2}$

10. (*6 pts*) Consider the following function

$$f(x) = \begin{cases} 2 + x^3 & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ 3\cos x & \text{if } x > 0 \end{cases}$$

Use the three step definition of continuity to determine if f is continuous at $x = 0$.

i) $f(0) = 3$

ii) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} 2 + x^3 = 2$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 3\cos x = 3$ $\Rightarrow \lim\limits_{x \to 0} f(x)$ D.N.E.

Hence, f is not continuous @ $x = 0$