Instr: Zeke Vogler | *November 18th, 2015*

Name: Key *Exam ID:*__________

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 8 pages, including this one.
2. **NO** calculators, books, notes, other written material, or help from other students allowed.
3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
4. You may NOT use L'Hopital's Rule to determine limits on this exam.
5. You may NOT use shortcuts from the textbook to determine limits to infinity on this exam.
6. You will be graded on proper use of limit, indeterminate form, and derivative notation.
7. You must put units on answers where units are appropriate.
8. Make sure to include graphs and sketches when it is part of the problem-solving process.
9. You have until 1:00pm to finish this exam.
10. Read the statement below and sign your name.

    I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ____________________________

"You can profit from your mistakes,
but that does not mean the more mistakes, the more profit." – Anonymous

GOOD LUCK!!!

1. (*12 pts*)

   (a) (*4 pts*) State the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   (b) (*8 pts*) Use the definition of the derivative to differentiate the function

$$f(x) = \sqrt{999x} \quad \text{(HINT: Let n = 999 be a constant)}$$

Let $n = 999 \Rightarrow f(x) = \sqrt{nx}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{n(x+h)} - \sqrt{nx}}{h} \cdot \frac{\sqrt{n(x+h)} + \sqrt{nx}}{\sqrt{n(x+h)} + \sqrt{nx}}$$

$$= \lim_{h \to 0} \frac{nx + nh - nx}{h(\sqrt{n(x+h)} + \sqrt{nx})} = \lim_{h \to 0} \frac{n\cancel{h}}{\cancel{h}(\sqrt{n(x+h)} + \sqrt{nx})}$$

$$= \frac{n}{2\sqrt{nx}} = \frac{999}{2\sqrt{999x}}$$ ∎

2. (*6 pts each*) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

(a) $g(x) = e^2 + \frac{1}{\sqrt{x}} + \log_7 x$ $= e^2 + x^{-\frac{1}{2}} + \log_7 x$

$\Rightarrow g'(x) = 0 - \frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{x}\frac{1}{\ln 7}$

D

(b) $f(x) = \frac{2^{-x}}{1 + \tan x}$

$\Rightarrow f'(x) = \frac{2^{-x}\ln 2(-1)(1+\tan x) - 2^{-x}(\sec^2 x)}{(1+\tan x)^2}$

D

(c) $y = x^{\cos x}$ $\Rightarrow \ln y = \ln x^{\cos x} = \cos x \cdot \ln x$

D

$\Rightarrow \frac{1}{y}y' = -\sin x \ln x + \cos x \frac{1}{x} \Rightarrow y' = y\left(-\sin x \ln x + \cos x \frac{1}{x}\right)$

$\Rightarrow y' = x^{\cos x}\left(-\sin x \ln x + \cos x \frac{1}{x}\right)$

3. (*8 pts*) Use linearization to estimate the value of $\sqrt{8}$. DO NOT SIMPLIFY THE ANSWER.

Let $f(x) = \sqrt{x}$ & $a = 9$.

$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

D

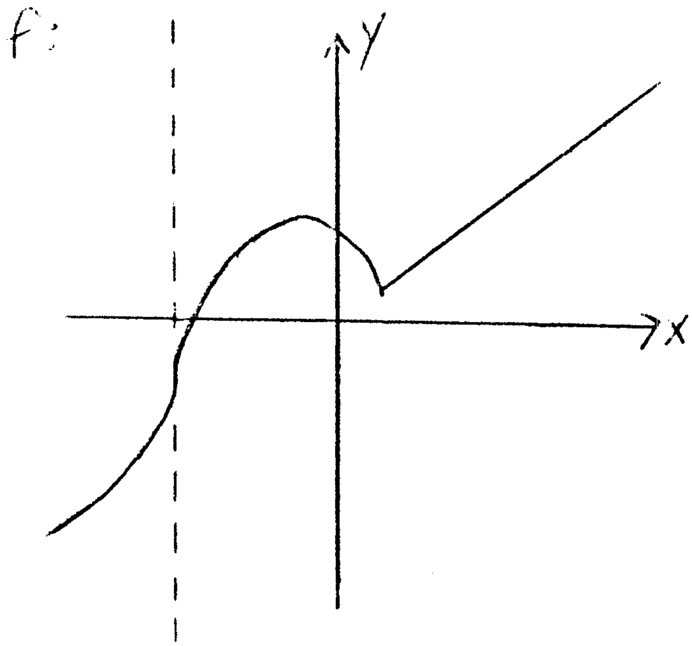
$L(x) = f(a) + f'(a)(x-a)$

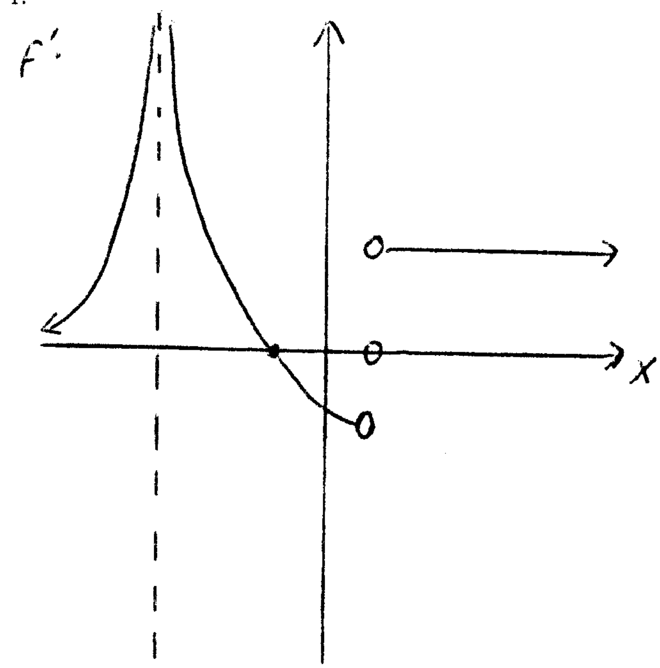
$= \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) = 3 + \frac{1}{6}(x-9)$

$\Rightarrow L(x) = 3 + \frac{1}{6}(x-9)$

Then, $\boxed{\sqrt{8} \approx L(8) = 3 + \frac{1}{6}(-1)}$ or $2\frac{5}{6}$

4. (*6 pts*) Sketch the graph of f' using the graph of f.





5. (*8 pts*) Consider the following equation for a curve

$$x^2 + y^3 = xy + 1$$

(a) (*6 pts*) Find the slope of the given curve at (0,1)

$D_x[x^2+y^3=xy+1] \Rightarrow 2x+3y^2y' = y+xy'$

$\Rightarrow 3y^2y' - xy' = y-2x \Rightarrow (3y^2-x)y' = y-2x$

$\Rightarrow y' = \frac{y-2x}{3y^2-x}$ @ $(0,1)$ $y' = \frac{1-0}{3(1)^2-0} = \frac{1}{3}$

(b) (*2 pts*) Find the equation of the normal line (i.e. perpendicular line) at (0,1) supposing you obtained in part (a)

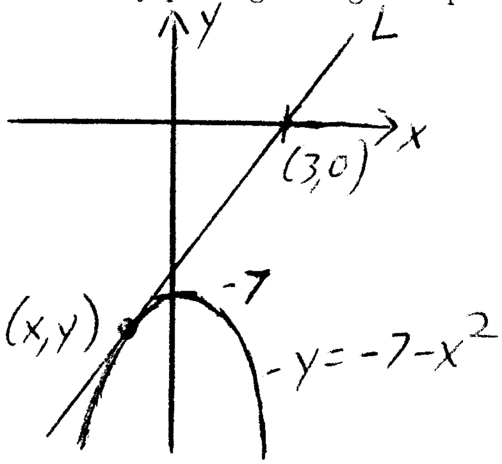
$$\frac{dy}{dx} = 3$$

$\Rightarrow \perp$ slope $= -\frac{1}{3}$

Equation of normal line: $y-1 = -\frac{1}{3}(x-0)$

$\Rightarrow \boxed{y = -\frac{1}{3}x+1}$

6. (*10 pts*) Find all points (x, y) on the graph $f(x) = -7 - x^2$ with tangent lines to the graph of f passing through the point $(3, 0)$.



Slope of line L is

1) $m = -2x$

2) $m = \frac{y-0}{x-3} = \frac{-7-x^2}{x-3} = -\frac{7+x^2}{x-3}$

Then,

$$-2x = m = -\frac{7+x^2}{x-3}$$

$\Rightarrow 2x(x-3) = 7+x^2 \Rightarrow 2x^2-6x = 7+x^2$

$\Rightarrow x^2-6x-7=0 \Rightarrow (x-7)(x+1)=0 \Rightarrow x=7,-1$

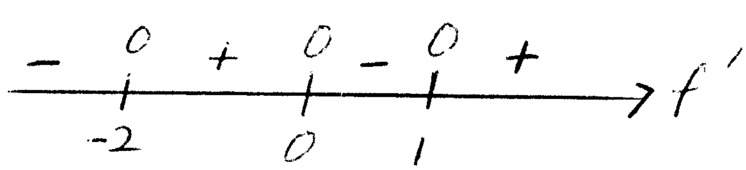
$f(7) = -7-7^2 = -56$; $f(-1) = -7-(-1)^2 = -8$

Pts: $(7,-56)$ or $(-1,-8)$

7. (*8 pts*) Let $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$. Solve $f'(x) = 0$ for x, and set up a sign chart for f'.

P $\Rightarrow f'(x) = x^3+x^2-2x = x(x^2+x-2) = x(x+2)(x-1) = 0$

$\Rightarrow x = 0, 1, -2$



8. (*10 pts*) Use the Intermediate Value Theorem to show that the equation $x^3 = 10 + \sqrt{x}$ is solvable. This is a writing exercise. You will be scored on proper style and mathematical correctness.

$x^3 = 10 + \sqrt{x} \Rightarrow x^3 - 10 - \sqrt{x} = 0$

Let $f(x) = x^3 - 10 - \sqrt{x}$ & $m = 0$

f is continuous for $x \geq 0$ since f is polynomial + square root.

$x = 0 \Rightarrow f(0) = 0 - 10 - 0 = -10 < 0$

$x = 4 \Rightarrow f(4) = 4^3 - 10 - \sqrt{4} > 0$

Choose interval $[0, 4]$ where $f(0) < m = 0 < f(4)$.

By IMVT there exists a c, $0 < c < 4$, such that $f(c) = m \Leftrightarrow c^3 - 10 - \sqrt{c} = 0 \Leftrightarrow c^3 = 10 + \sqrt{c}$

Hence, equation is solvable.

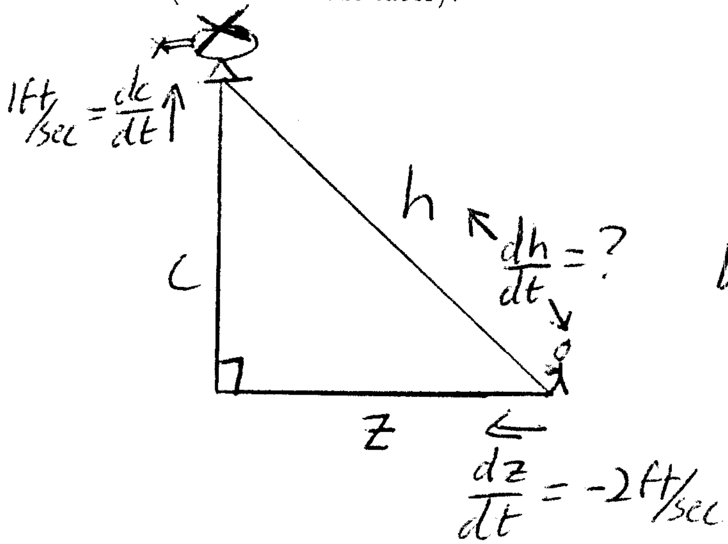
9. (*8 pts*) Let $f(x) = x + e^x$. Compute $D\{f^{-1}(1)\}$. Note that $f(0) = 1$.

$f(x) = x + e^x \overset{D}{\Rightarrow} f'(x) = 1 + e^x$; $f(0) = 1 \Rightarrow f^{-1}(1) = 0$

$D\{f^{-1}(x)\} = \dfrac{1}{f'(f^{-1}(x))} \Rightarrow D\{f^{-1}(1)\} = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)}$

$= \dfrac{1}{1 + e^0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

10. (*12 pts*) Claire Redfield is escaping the Umbrella company on a helicopter. The helicopter is on the helipad and rising at a rate of 1 ft/sec. There is also a Zombie Master on the helipad which is 10 feet from Claire. The Zombie Master is moving towards Claire (in a straight line) at a rate of 2 ft/sec Feeling guilty of leaving the Zombie Master alive, Claire starts pulling out her rocket launcher. It takes Claire 3 seconds to get the rocket launcher ready. How fast is the distance between her (on the helicopter) and the Zombie Master increasing at this time (i.e. 3 seconds later)?



Goal: Find $\frac{dh}{dt}$ @ $t=3$

$$c^2 + z^2 = h^2$$

$$D_t[c^2 + z^2 = h^2]$$

$$\Rightarrow \cancel{2}c\frac{dc}{dt} + \cancel{2}z\frac{dz}{dt} = \cancel{2}h\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{c\frac{dc}{dt} + z\frac{dz}{dt}}{h}$$

For c: Initially 0 ft
After 3 sec @ $1\,ft/sec$
$\Rightarrow$ moves 3 ft
$\Rightarrow c = 3\,ft$

For z: Initially 10 ft
After 3 sec @ $-2\,ft/sec$
$\Rightarrow$ moves -6 ft
$\Rightarrow z = 10 - 6 = 4\,ft$

For h: $h^2 = c^2 + z^2$
$\Rightarrow h^2 = 9 + 16 = 25$
$\Rightarrow h = 5\,ft$

$$\frac{dh}{dt} = \frac{3 \cdot 1 + 4(-2)}{5} = \frac{-5}{5}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = -1\,ft/sec}$$