1.) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = \sin^5(3-x)$

$\xrightarrow{D} y' = 5\sin^4(3-x)\cdot\cos(3-x)\cdot(-1)$

b.) $f(x) = \dfrac{2^x}{6+e^{3x}}$

$\xrightarrow{D} f'(x) = \dfrac{(6+e^{3x})\cdot 2^x \ln 2 - 2^x\cdot e^{3x}\cdot 3}{(6+e^{3x})^2}$

c.) $f(x) = (\ln x)^3 \cdot \log_4(\tan x)$

$\xrightarrow{D} f'(x) = (\ln x)^3 \cdot \frac{1}{\tan x}\cdot \sec^2 x \cdot \frac{1}{\ln 4} + 3(\ln x)^2 \cdot \frac{1}{x}\cdot \log_4(\tan x)$

d.) $f(x) = (\ln x)^x$ $\rightarrow \ln f(x) = \ln(\ln x)^x = x\cdot\ln(\ln x) \xrightarrow{D}$

$\frac{1}{f(x)}\cdot f'(x) = \cancel{x}\cdot\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}} + (1)\cdot\ln(\ln x) \longrightarrow$

$f'(x) = (\ln x)^x \cdot \left\{\frac{1}{\ln x} + \ln(\ln x)\right\}$

5.) Consider the function $f(x) = x - \sqrt{x}$ defined on the closed interval $[0, 4]$. Verify that f satisfies the assumptions of the Mean Value Theorem (MVT) and find all values of c guaranteed by the MVT.

$f(x) = x - \sqrt{x}$ is cont. on the closed interval $[0,4]$ since it is the difference of cont. functions; $f'(x) = 1 - \frac{1}{2\sqrt{x}}$ so f is diff. on the open interval $(0,4)$. f' by MVT there is a # c, $0<c<4$, satisfying $\dfrac{f(4)-f(0)}{4-0} = f'(c) \rightarrow$

$\dfrac{2-0}{4-0} = 1 - \dfrac{1}{2\sqrt{c}} \longrightarrow \dfrac{1}{2\sqrt{c}} = \dfrac{1}{2} \longrightarrow \sqrt{c} = 1 \rightarrow$

$\boxed{c = 1}$

48

3.) The manager of the Economy Motel charges \$30 per room and rents ~~50~~ rooms each night. For each \$5 increase in room charge four (4) fewer rooms are rented. What charge per room will maximize the total amount of money the manager will make in one night ?

Let x : # of \$5 increases,

$$\text{max. } T = \underset{\substack{\uparrow \\ \text{charge per room}}}{(30+5x)}\underset{\substack{\uparrow \\ \text{\# of rooms}}}{(\overset{48}{}-4x)} \rightarrow$$

$$T' = (30+5x)(-4) + (5)(\overset{48}{}-4x)$$

$$= -120 - 20x + \underset{240}{} - 20x$$

$$= 120 - 40x = 0$$

$$\overset{+ \quad 0 \quad -}{\overline{\qquad\quad \underset{x=3}{|} \qquad\quad}} \; T'$$

charge : \$45

rooms : 36

\$: 1620

4.) Use the limit definition of derivative to differentiate $f(x) = \dfrac{x^2}{x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)^2}{x+h+1} - \dfrac{x^2}{x+1}}{\dfrac{h}{1}}$$

$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)(x+1)-(x+h+1)x^2}{(x+h+1)\ (x+1)} \cdot \frac{1}{h}$$

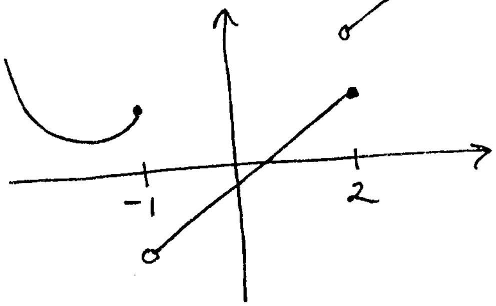
$$= \lim_{h\to 0} \frac{\cancel{x^3}+2hx^2+h^2x+\cancel{x^2}+2hx+h^2-\cancel{x^3}-hx^2-\cancel{x^2}}{(x+h+1)(x+1)\ \cdot\ h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x^2+hx+2x+h-x^2)}{(x+h+1)\ (x+1)\cdot \cancel{h}}$$

$$= \frac{x^2+2x}{(x+1)^2}$$

5.) Use limits to determine the values of the constants A and B so that the following function is continuous for all values of x .

$$f(x) = \begin{cases} Bx^2 + Ax, & \text{if } x \le -1 \\ 2B - Ax, & \text{if } -1 < x \le 2 \\ x + 3, & \text{if } x > 2 \ . \end{cases}$$



$$\lim_{x \to -1^-} (Bx^2 + Ax) = \lim_{x \to -1^+} (2B - Ax)$$

$$\rightarrow B - A = 2B + A \rightarrow \boxed{B = -2A} \ ;$$

$$\lim_{x \to 2^-} (2B - Ax) = \lim_{x \to 2^+} (x + 3)$$

$$\rightarrow \boxed{2B - 2A = 5} \longrightarrow 2(-2A) - 2A = 5 \rightarrow$$

$$-6A = 5 \rightarrow \boxed{A = -5/6} \ , \ \boxed{B = 5/3}$$

7.) Use a linearization to estimate the value of $\sqrt{68}$.

Let $f(x) = \sqrt{x}$ and $a = 64$, then $f'(x) = \frac{1}{2\sqrt{x}}$

and $L(x) = f(a) + f'(a)(x - a)$

$$= \sqrt{64} + \frac{1}{2\sqrt{64}}(x - 64) = 8 + \frac{1}{16}(x - 64) = 8 + \frac{1}{16}x - 4$$

$$\rightarrow L(x) = 4 + \frac{1}{16}x \ ; \text{ then}$$

$$\sqrt{68} \approx L(68) = 4 + \frac{1}{16}(68) = 8.25$$

7.) The radius of a sphere is measured with an absolute percentage error of at most 4% . Use differentials to estimate the maximum absolute percentage error in computing the volume of the sphere. ($V = \frac{4}{3}\pi r^3$.)

$V' = 4\pi r^2$ and $\frac{|\Delta r|}{r} \leq 4\%$, estimate

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta r|}{V} = \frac{|\cancel{4}\cancel{\pi}\cancel{r^2} \cdot \Delta r|}{\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}}$$

$$= 3\frac{|\Delta r|}{r} \leq 3(4\%) = 12\% .$$

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8. Use the Intermediate Value Theorem to show that the equation $x^3 + x = \sqrt{x+4}$ is solvable. This is a writing exercise.

$$x^3 + x = \sqrt{x+4} \iff x^3 + x - \sqrt{x+4} = 0$$

Let $f(x) = x^3 + x - \sqrt{x+4}$ & $m = 0$

f is cont. (poly. + sqrt. is cont.)

Since $f(0) = -2 < 0$ & $f(5) = 5^3 + 5 - 3 > 0$

Choose interval $[0, 5]$ since $f(0) < m = 0 < f(5)$

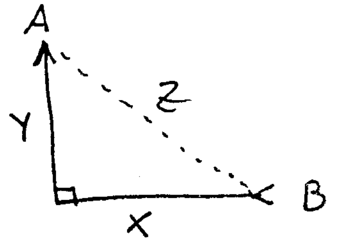
By IMVT, there exists @ least 1 c b/w 0 & 5

s.t. $f(c) = 0 \iff c^3 + c = \sqrt{c+4}$

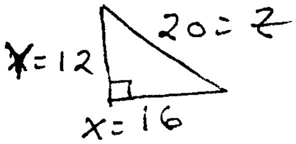
Hence, equation is solvable.

9.) Car B is 34 miles directly east of car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph.

a.) At what rate is the distance between the cars changing after t = $\frac{1}{5}$ hr. ?



$\frac{dY}{dt} = 60$ mph, $\frac{dx}{dt} = -90$ mph, find $\frac{dz}{dt}$ when $t = \frac{1}{5}$ hr → $x = 12$ mi,
$Y = 16$ mi. :



$$x^2 + Y^2 = z^2 \rightarrow \cancel{2}x\frac{dx}{dt} + \cancel{2}Y\frac{dY}{dt} = \cancel{2}z\frac{dz}{dt}$$

$$\rightarrow (16)(-9\cancel{0}) + (12)(6\cancel{0}) = (2\cancel{0})\frac{dz}{dt}$$

$$\rightarrow \frac{dz}{dt} = -36 \text{ mph}$$

9.) Consider the Beverton-Holt Recursion given by $N_{t+1} = \dfrac{120N_t}{100 + N_t}$ with initial amount $N_0 = 10$.

a.) Find all fixed points for this recursion.

$$L = \frac{120L}{100+L} \rightarrow L(100+L) = 120L \rightarrow$$

$$L^2 + 100L - 120L = 0 \rightarrow L^2 - 20L = 0 \rightarrow$$

$$L(L-20) = 0 \rightarrow L = 0, \ L = 20$$

b.) Determine the growth parameter R and the carrying capacity K.

$$N_{t+1} = \frac{120N_t}{100+N_t}\cdot\frac{1/100}{1/100} = \frac{1.2N_t}{1+\frac{1}{100}N_t} \quad \text{so } R = 1.2$$

and from part a.) $K = 20$

c.) Find N_1.

$$N_1 = \frac{120N_0}{100+N_0} = \frac{120(10)}{100+10} \approx 10.91$$

10.) Find the slope and concavity of the graph $xy + y^2 = 3x + 1$ at the point $(0, -1)$.

$$\xrightarrow{D} xy' + y + 2yy' = 3 \rightarrow y' = \frac{3-y}{x+2y}$$

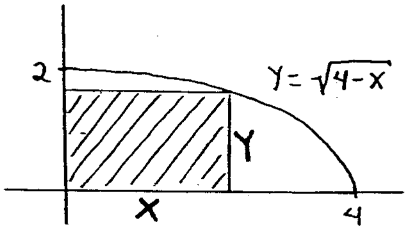
and $x=0, y=-1 \rightarrow y' = \frac{4}{-2} = \boxed{-2 = \text{slope}}$;

$$y'' = \frac{(x+2y)(-y') - (3-y)(1+2y')}{(x+2y)^2}$$

$$= \frac{(-2)(2) - (4)(-3)}{(-2)^2} = \boxed{2 = y''}$$ so

concave up.

11.) Consider all rectangles in the first quadrant inscribed in such a way that their bases lie on the x-axis with the top corner on the graph of $y = \sqrt{4-x}$. Find the length and width of the rectangle of maximum area.



max. area

$$A = xy = x\sqrt{4-x} \longrightarrow$$

$$A' = x \cdot \tfrac{1}{2}(4-x)^{-1/2}(-1) + \sqrt{4-x}$$

$$= \frac{-x}{2\sqrt{4-x}} + \frac{\sqrt{4-x}}{1} = \frac{-x + 2(4-x)}{2\sqrt{4-x}}$$

$$= \frac{8-3x}{2\sqrt{4-x}} = 0$$

$+ \quad 0 \quad -$ $\quad A'$

$x = 8/3$

$A = \frac{16}{3\sqrt{3}}$ $\quad y = \frac{2}{\sqrt{3}}$

12.) Consider the function $f(x) = x\,e^{(\frac{-x}{2})}$. Determine where f is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, x- and y-intercepts, and vertical and horizontal asymptotes. Sketch the graph. You may assume that $f'(x) = (1 - \frac{x}{2})\,e^{(\frac{-x}{2})}$ and $f''(x) = (\frac{x}{4} - 1)\,e^{(\frac{-x}{2})}$.

$+ \quad 0 \quad -$ $\quad f'$

$- \quad 0 \quad +$ $\quad f''$

abs. max. $\begin{cases} X=2 \\ Y=\frac{2}{e} \end{cases}$

infl. pt. $\begin{cases} X=4 \\ Y=\frac{4}{e^2} \end{cases}$

$X=0 : Y=0$ and $Y=0 : X=0$

f is ↑ for $X<2$

f is ↓ for $X>2$

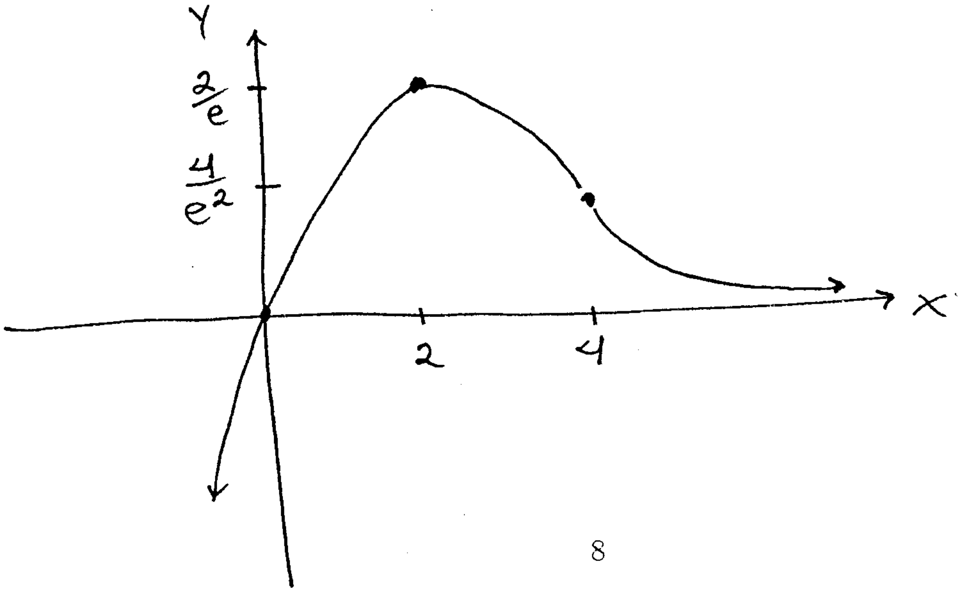
f is ∪ for $X>4$

f is ∩ for $X<4$

$$\lim_{X\to-\infty} x e^{-\frac{x}{2}} = -\infty \cdot \infty = -\infty$$

$$\lim_{X\to+\infty} x e^{-\frac{x}{2}} = \lim_{X\to+\infty} \frac{x}{e^{x/2}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X\to+\infty} \frac{1}{\frac{1}{2}e^{x/2}} = \text{"}\frac{1}{\infty}\text{"} = 0$$

so horizontal asymptote $Y=0$



4.) In 1947 earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis showed that the scrolls contained 76 % of their original carbon-14. Assuming the half-life of carbon-14 is 5730 years, estimate the age of the Dead Sea Scrolls when they were found.

assume $N = N_0 e^{kt}$, then $t = 5730$ yrs, $N = \frac{1}{2} N_0$

$$\rightarrow \frac{1}{2}\cancel{N_0} = \cancel{N_0} e^{5730k} \rightarrow \ln(1/2) = \ln e^{5730k} = 5730k$$

$$\rightarrow k = \frac{\ln(1/2)}{5730} \rightarrow N = N_0 e^{\frac{\ln(1/2)}{5730}t};$$ then

$$N = 0.76 N_0 \rightarrow 0.76 \cancel{N_0} = \cancel{N_0} e^{\frac{\ln(1/2)}{5730}t} \rightarrow$$

$$\ln(0.76) = \ln e^{\frac{\ln(1/2)}{5730}t} = \frac{\ln(1/2)}{5730} t \rightarrow$$

$$t = \frac{5730 \cdot \ln(0.76)}{\ln(1/2)} \approx 2268.7 \text{ yrs.}$$

14.) Evaluate the following limits.

a.) $$\lim_{x\to 0} \frac{x \sin x}{(\arctan x)^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{x \cos x + \sin x}{2 \arctan x \cdot \frac{1}{1+x^2}}$$

$$= \lim_{x \to 0} \frac{(x \cos x + \sin x)(x^2+1)}{2 \arctan x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{(x \cos x + \sin x)(2x) + (-x \sin x + \cos x + \cos x)(x^2+1)}{2 \cdot \frac{1}{1+x^2}}$$

$$= \frac{2}{2} = 1$$

b.) $\lim_{x\to 0+} x \ln x$ $=$ "$0 \cdot -\infty$" $= \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$

$$\overset{\text{"}\frac{-\infty}{\infty}\text{"}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

c.) $\lim_{x\to\infty} (x^3+4)^{\frac{1}{x}}$ $=$ "∞^0"

$$\ln\left(\lim_{x\to\infty} (x^3+4)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln (x^3+4)^{1/x}$$

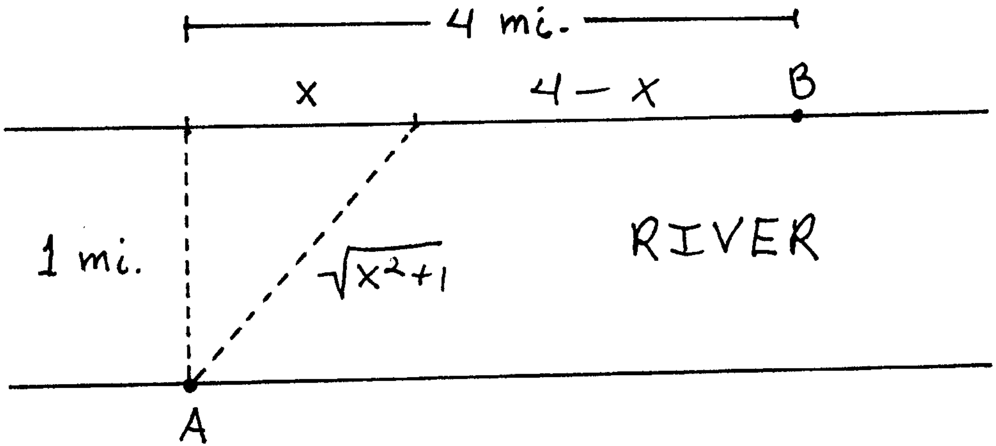
$$= \lim_{x\to\infty} \frac{\ln(x^3+4)}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+4}}{1}$$

$$= \lim_{x\to\infty} \frac{3x^2 \cdot \frac{1}{x^2}}{(x^3+4)\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3}{x+\frac{4}{x^2}}$$

$$= \text{"}\frac{3}{\infty+0}\text{"} = 0 \quad \text{so } \lim_{x\to\infty} (x^3+4)^{1/x} = 1$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) You are standing at point A on the edge of a river 1 mile wide. You are to get to point B, which is 4 miles from the point directly across the river from you. You can paddle a canoe in the water at a speed of 10 miles per hour and you can ride a bicycle on land at a speed of 15 miles per hour. Determine x so that the time it takes to go from point A to point B is a minimum and determine the minimum time.



$D = RT$ so $T = D/R$; minimize time

$T = T_{water} + T_{land} \rightarrow$

$$T = \frac{\sqrt{x^2+1}}{10} + \frac{4-x}{15} \xrightarrow{D}$$

$$T' = \frac{1}{10} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - \frac{1}{15}$$

$$= \frac{x}{10\sqrt{x^2+1}} - \frac{1}{15} = 0 \rightarrow \frac{x}{10\sqrt{x^2+1}} = \frac{1}{15} \rightarrow$$

$15x = 10\sqrt{x^2+1} \rightarrow 225x^2 = 100(x^2+1) \rightarrow$

$225x^2 = 100x^2 + 100 \rightarrow 125x^2 = 100 \rightarrow$

$x^2 = \frac{100}{125} \rightarrow x = \sqrt{\frac{100}{125}} \approx 0.894$ mi

$-\ 0\ +$ T'

$x = \sqrt{\frac{100}{125}}$ mi , min $T \approx 0.34$ hr.