Math 17A
Vogler
Recursions, Sequences, Fixed Points, and Limits

EXAMPLE : The following recursions and inital values determine a sequence. Find a_n for $n = 1, 2, 3, 4, 5$.

1.) $a_{n+1} = 2a_n + 3, a_0 = -1$

$$\begin{aligned} a_1 &= 2a_0 + 3 = 2(-1) + 3 = 1, \\ a_2 &= 2a_1 + 3 = 2(1) + 3 = 5, \\ a_3 &= 2a_2 + 3 = 2(5) + 3 = 13, \\ a_4 &= 2a_3 + 3 = 2(13) + 3 = 29, \\ a_5 &= 2a_4 + 3 = 2(29) + 3 = 61 . \end{aligned}$$

Hence $\lim_{n \to \infty} a_n = \infty$ (DNE) .

2.) $a_{n+1} = 2a_n + 3, a_0 = -3$

$$\begin{aligned} a_1 &= 2a_0 + 3 = 2(-3) + 3 = -3, \\ a_2 &= 2a_1 + 3 = 2(-3) + 3 = -3, \\ a_3 &= 2a_2 + 3 = 2(-3) + 3 = -3, \\ a_4 &= 2a_3 + 3 = 2(-3) + 3 = -3, \\ a_5 &= 2a_4 + 3 = 2(-3) + 3 = -3 . \end{aligned}$$

Hence $\lim_{n \to \infty} a_n = -3$.

DEFINITION : Let $a_{n+1} = f(a_n)$, $a_0 = L$, for $n = 1, 2, 3, 4, \cdots$ be a recursion and initial value which determines a sequence. The initial value L is called a fixed point for the recursion if all successive values of a_n are equal to L, i.e., if $L = f(L)$.

NOTE :

I.) The number -3 is a fixed point for the previous example.

II.) The initial value is sometimes critical in determining if the sequence converges or diverges.

III.) A fixed point represents a potential limit for the sequence generated by the recursion and its initial value.

IV.) Every limit of an associated sequence is a fixed point for the recursion.

EXAMPLE : Find all fixed points for each recursion.

1.) $a_{n+1} = (1/2)a_n - (3/4)$

2.) $a_{n+1} = \dfrac{2}{a_n - 1}$

3.) $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 12}$

$$a_{n+1} = \frac{2}{a_n - 1}$$

n	a{n}			n	a{n}			n	a{n}	
0	5.0000	0.5000		0	−10.0000	−0.1818		0	2.0000	2.0000
1	0.5000	−4.0000		1	−0.1818	−1.6923		1	2.0000	2.0000
2	−4.0000	−0.4000		2	−1.6923	−0.7429		2	2.0000	2.0000
3	−0.4000	−1.4286		3	−0.7429	−1.1475		3	2.0000	2.0000
4	−1.4286	−0.8235		4	−1.1475	−0.9313		4	2.0000	2.0000
5	−0.8235	−1.0968		5	−0.9313	−1.0356		5	2.0000	2.0000
6	−1.0968	−0.9538		6	−1.0356	−0.9825		6	2.0000	2.0000
7	−0.9538	−1.0236		7	−0.9825	−1.0088		7	2.0000	2.0000
8	−1.0236	−0.9883		8	−1.0088	−0.9956		8	2.0000	2.0000
9	−0.9883	−1.0059		9	−0.9956	−1.0022		9	2.0000	2.0000
10	−1.0059	−0.9971		10	−1.0022	−0.9989		10	2.0000	2.0000
11	−0.9971	−1.0015		11	−0.9989	−1.0005		11	2.0000	2.0000
12	−1.0015	−0.9993		12	−1.0005	−0.9997		12	2.0000	2.0000
13	−0.9993	−1.0004		13	−0.9997	−1.0001		13	2.0000	2.0000
14	−1.0004	−0.9998		14	−1.0001	−0.9999		14	2.0000	2.0000
15	−0.9998	−1.0001		15	−0.9999	−1.0000		15	2.0000	2.0000
16	−1.0001	−1.0000		16	−1.0000	−1.0000		16	2.0000	2.0000
17	−1.0000	−1.0000		17	−1.0000	−1.0000		17	2.0000	2.0000
18	−1.0000	−1.0000		18	−1.0000	−1.0000		18	2.0000	2.0000
19	−1.0000	−1.0000		19	−1.0000	−1.0000		19	2.0000	2.0000
20	−1.0000	−1.0000		20	−1.0000	−1.0000		20	2.0000	2.0000
21	−1.0000	−1.0000		21	−1.0000	−1.0000		21	2.0000	2.0000
22	−1.0000	−1.0000		22	−1.0000	−1.0000		22	2.0000	2.0000
23	−1.0000	−1.0000		23	−1.0000	−1.0000		23	2.0000	2.0000