Math 17A
Vogler

Exploring the Limit of a Function

Consider the function $f(x) = \dfrac{\sqrt[3]{x} - 1}{1 - \sqrt{x}}$. Notice $f(1) = \text{'}\frac{0}{0}\text{'}$ is not defined.

Q: Although the function is not defined at $x=1$, it is well defined for x-values close to 1. Can we study the behavior of f at $x=1$ without using $f(1)$?

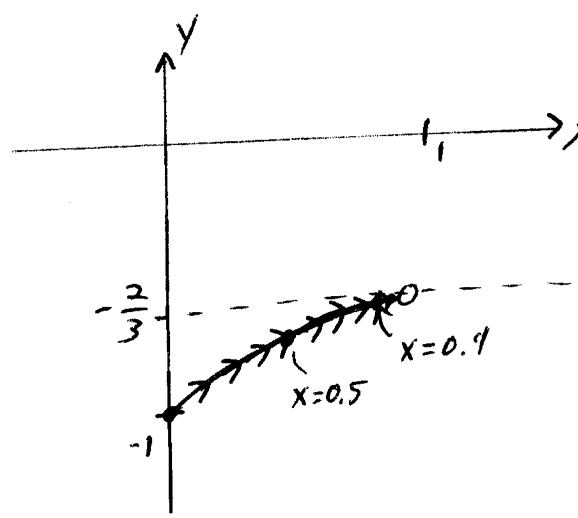
A: Yes. By studying the behavior of f at values close to $x=1$.

Approaching $x=1$ from the left $(x<1)$

Using a calculator, we can build the following table:

x	$y=f(x)$
0	-1
0.5	-0.704350
0.9	-0.672503
0.99	-0.667225
0.999	-0.666722
0.9999	-0.666672
⋮	⋮
1	?

This data is shown on the graph $\Longrightarrow$



Appears that when approaching $x=1$ from left $(x<1)$, the function is approaching a finite value, in particular $y = -\frac{2}{3}$

This practice leads to the following:

Defn: $\lim_{x \to a^-} f(x)$ when x approaches a from left $(x<a)$

Ex The above table leads us to believe

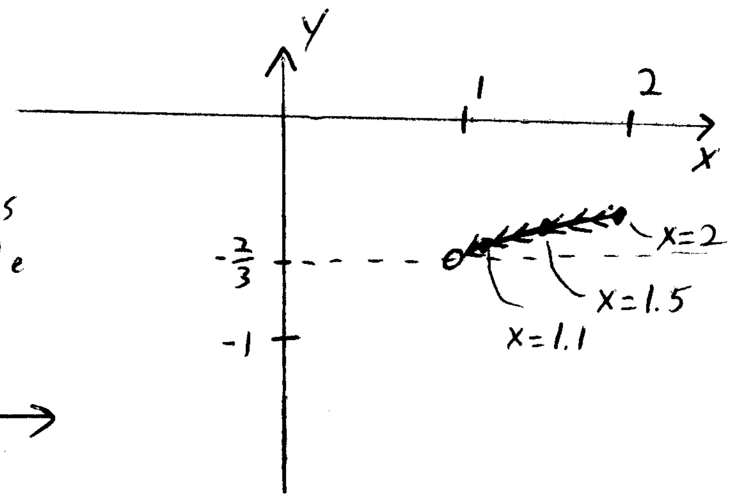
$$\lim_{x \to 1^-} \frac{\sqrt[3]{x} - 1}{1 - \sqrt{x}} = \frac{-2}{3}$$

Approaching $x=1$ from the right $(x>1)$

Using a calculator, we can build the following table

x	$y=f(x)$
2	-0.627505
1.5	-0.643905
1.1	-0.661358
1.01	-0.666114
1.001	-0.666611
1.0001	-0.666661
1.00001	-0.666666

This data is shown on the graph $\longrightarrow$



Again, it appears that when approaching $x=1$ from the right $(x>1)$, the function is approaching $y=-\frac{2}{3}$.

This practice leads to the following:

Defn $\lim\limits_{x\to a^+} f(x)$ when x approaches a from right $(x>a)$

Ex The above table leads us to believe

$$\lim_{x\to 1^+} \frac{\sqrt[3]{x}-1}{1-\sqrt{x}} = -\frac{2}{3}$$

Approaching $x=1$ from either direction

We argued $\lim\limits_{x\to 1^+} \frac{\sqrt[3]{x}-1}{1-\sqrt{x}} = \lim\limits_{x\to 1^-} \frac{\sqrt[3]{x}-1}{1-\sqrt{x}} = -\frac{2}{3}$

This happens often enough, we wish to have a concept to describe it. Namely, $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{1-\sqrt{x}} = -\frac{2}{3}$ ($\lim\limits_{x\to 1}$ means approaching from both directions)

This leads to:

Defn (Informal) Let $y=f(x)$ be a function. The limit as x approaches a number 'a' of $f(x)$ equals L, written $\lim\limits_{x\to a} f(x)=L$, means as x-values get closer to 'a', the corresponding y-values get closer to L.

Notes: 1) x approaches 'a' from both sides ($\lim\limits_{x\to a^-}$ & $\lim\limits_{x\to a^+}$)

2) L must be finite & unique. Also, $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$.

3) L is an 'expected' y-value.

4) If $L=\pm\infty$, we say limit does not exist (DNE)