Vogler
Math 125A

Connected and Path-Connected

Defn Let $\gamma: \mathbb{R} \to \mathbb{R}^n$ be continuous. We call γ a path and its image $\gamma(\mathbb{R}) \subseteq \mathbb{R}^n$ a curve.

Prop (21.2)
If $f_1, f_2, \ldots, f_n$ are continuous functions on $\mathbb{R}$, then
$$\gamma(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$
is a path (i.e. is continuous) in $\mathbb{R}^n$.

- Note: This gives us a quick way of determining continuous functions into $\mathbb{R}^n$ by checking each coordinate function for continuity.

Defn Let $E \subseteq S_1$. E is disconnected if there exists disjoint open subsets $U_1, U_2 \subseteq S_1$ such that
$$E \subseteq U_1 \cup U_2, \; E \cap U_1 \neq \emptyset, \text{ and } E \cap U_2 \neq \emptyset$$
A set E is connected if it is not disconnected.

- Note: U_1 and U_2 are disjoint if $U_1 \cap U_2 = \emptyset$ (i.e. empty set).

Prop
All intervals in $\mathbb{R}$ are connected. Also, all neighborhoods in any metric space are connected.

Thm (22.2)
Let $f: S_1 \to S_2$ be continuous. If $E \subseteq S_1$ is connected, then $f(E) \subseteq S_2$ is also connected.
- Moral: Continuous functions preserve connectedness.

Cor (22.3)
Let $f: S_1 \to \mathbb{R}$ be continuous. If $E \subseteq S_1$ is connected, then $f(E) \subseteq \mathbb{R}$ is an interval.
- Notes: 1) This f has the intermediate value property.
2) This fact is the generalization of the Intermediate Value Theorem on an arbitrary metric space. Also, this tells us that connected sets are the generalization of intervals on an arbitrary metric space.

Defn A set $E \subseteq S_1$ is path-connected if for all $s, t \in E$ there exists a continuous function $\gamma: [a,b] \to E$ such that $\gamma(a) = s$ and $\gamma(b) = t$.

Thm (22.5)
If $E \subseteq S_1$ is path-connected, then E is connected.
- Moral: Path-connected is a 'stronger' notion than connected.

Defn Let $S \subseteq \mathbb{R}$. Let $C(S)$ be the set of all bounded continuous real-valued functions on S and, for $f, g \in C(S)$, define
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}.$$
- Notes: 1) One can easily show $(C(S), d)$ is a metric space.
2) A sequence of functions $\{f_n\} \subseteq C(S)$ converges to $f \in C(S)$ in this space is equivalent to $\{f_n\} \rightrightarrows f$ (i.e. uniform convergence).