Math 125A
Vogler

Continuity

Note! For this class, f is a function on $\mathbb{R}$ and x is in the domain of f, unless otherwise stated.

__Defn__ f is __continuous at x_0__ if for all sequences $\{x_n\}$ converging to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. f is called __continuous__ if it's continuous at x_0 for each x_0 in it's domain. If f is not continuous at x_0, we say it's __discontinuous at x_0__.

-Notes: 1) The sequence $\{x_n\}$ leads to a corresponding one $\{f(x_n)\}$, & continuity can be described by

$$\forall \{x_n\} \longrightarrow x_0 \implies \{f(x_n)\} \longrightarrow f(x_0).$$

2) I might describe this definition as the 'sequential form of continuity'.

3) $\lim_{n\to\infty} f(x_n) = f(x_0) = f(\lim_{n\to\infty} x_n)$, i.e. continuous fns let limits 'pass through' them.

-Moral: For continuous fns, approaching a point of interest is the same as being at that point.

__Thm (17.2)__ f is continuous at x_0 if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \text{ with } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon. \quad (1)$$

-Notes: 1) I will refer to (1) as the '$\varepsilon\delta$-property'.

2) Some text books use the $\varepsilon\delta$-property as the definition of continuity which is equivalent to ours b/c of Thm.

3) The negation of (1) is

$\exists \varepsilon > 0, \forall \delta > 0$, such that $\exists x$ with $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \geq \varepsilon$

- Moral: For continuous fns, the inputs being arbitrarily close imply the outputs will be close.

<u>Thm</u> (17.3 – 17.5)

Let f and g be continuous at x_0. Then

1) $|f|$ is continuous at x_0.
2) kf $\forall k \in \mathbb{R}$ is continuous at x_0.
3) $f+g$ is continuous at x_0.
4) $f \cdot g$ is continuous at x_0.
5) $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.
6) $g \circ f$ is continuous at x_0 if g is continuous at $f(x_0)$.

- Moral: Functional composition and algebra preserve continuity as long as something doesn't go 'wrong' (like division by $\emptyset$)

<u>List of continuous functions</u>

1) $\sin x$
2) $\cos x$
3) e^x
4) 2^x
5) $\log_b x$ for any base b
6) x^p $\forall p \in \mathbb{R}$

Notes: 1) These functions are only continuous on their respective domains.

2) With this list & above theorem, you can prove most of the functions you've encountered in your math career are continuous (See HW # 17.3)