Math 125A
Vogler

Limits of Functions

Defn Let $a \in \mathbb{R}$ and $L \in \mathbb{R} \cup \{\pm\infty\}$.

a) The limit of f as x approaches a equals L, written $\lim_{x \to a} f(x) = L$, if for all sequences $\{x_n\} \subseteq (c,a) \cup (a,b)$ with $\{x_n\} \to a$, $\lim_{n \to \infty} f(x_n) = L$.

b) The left-handed limit $\lim_{x \to a^-} f(x) = L$ if for all sequences $\{x_n\} \subseteq (c,a)$ with $\{x_n\} \to a$, $\lim_{n \to \infty} f(x_n) = L$.

c) The right-handed limit $\lim_{x \to a^+} f(x) = L$ if for all sequences $\{x_n\} \subseteq (a,b)$ with $\{x_n\} \to a$ $\lim_{n \to \infty} f(x_n) = L$.

d) $\lim_{x \to \infty} f(x) = L$ if for all sequences $\{x_n\} \subseteq (c,\infty)$ with $\{x_n\} \to \infty$, $\lim_{n \to \infty} f(x_n) = L$.

e) $\lim_{x \to -\infty} f(x) = L$ if for all sequences $\{x_n\} \subseteq (-\infty,b)$ with $\{x_n\} \to -\infty$, $\lim_{n \to \infty} f(x_n) = L$.

-Notes: 1) These definitions allow us to make formal these concepts we learned in math 21A using concepts (ie. limits of sequences) we studied earlier in the book.
2) Obviously, $b, c \in \mathbb{R}$ and all intervals MUST be in domain of f.
3) If any of these limits exists, there are unique and independent of the choice of interval (ie. does not depend on choice of b,c).

Thm (20.4, 20.5)

Let f and g be functions where $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ exist and are finite. Then

1) $\lim_{x \to a} (f+g)(x) = L_1 + L_2$
2) $\lim_{x \to a} (f \cdot g)(x) = L_1 \cdot L_2$
3) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}$ as long as $g(x_n) \neq 0$ $\forall n$ and $L_2 \neq 0$.
4) If g is defined on range of f and f is continuous $\lim_{x \to a} (g \circ f)(x) = g(L)$, provided g is also defined at L.

Thm (20.6)

$\lim_{x \to a} f(x) = L$ for L finite if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x$ with $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. (1)

- Notes: 1) I'll refer to (1) as the '$\varepsilon\delta$-property' for limits.
2) Many textbooks, included our math 21A book, use (1) as the definition of the limit, which is equivalent to ours because of this theorem.
3) For left-handed limits, $\lim_{x \to a^-} f(x) = L$, this becomes $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x$ with $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$.
4) For right-handed limits $\lim_{x \to a^+} f(x) = L$, this becomes $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x$ with $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.

Thm (20.10)

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$

- Note: I will refer to the right hand side as the 'kiss condition'.
- Moral: A limit existing is equivalent to both the left and right limits existing and matching.