Math 125A
Vogler

Continuity on Metric Spaces

Note: For this class, unless otherwise stated, we have the following assumptions:

1) $\mathbb{R}^n$ with $n \in \mathbb{N}$ is the metric space with the Euclidean metric $d(\vec{x},\vec{y}) = \left[\sum_{k=1}^{n} (x_k - y_k)\right]^{1/2}$ where $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$. For $\mathbb{R}$, $d(x,y) = |x-y|$.
2) The set S_i with metric d_i, denoted (S_i, d_i), is a metric space for all $i \in \mathbb{N}$.

Defn Let $x_0 \in S_1$ and $r > 0$. The (open) neighborhood around x_0 with radius r, denoted $B_r(x_0)$, is the set

$$B_r(x_0) := \{x \in S_1 : d_1(x, x_0) < r\}$$

-Notes: 1) Some people refer to this set as an open ball.
2) These neighborhoods can be viewed as a basis for all open sets in S_1 and are the fundamental open sets in the metric space (S_1, d_1).

Defn A function $f: S_1 \to S_2$ is continuous at $x_0 \in S_1$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in S_1$ with $d_1(x, x_0) < \delta$ $\Rightarrow d_2(f(x), f(x_0)) < \varepsilon$.

f is uniformly continuous on $E \subseteq S_1$ if
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in S_1$ with $d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

-Notes: 1) When $S_1 = S_2 = \mathbb{R}$, these definitions match our previous ones. So these are generalizations of continuity to arbitrary metric spaces.

2) We can recast these definitions using neighborhoods as follows: f is <u>continuous</u> at $x_0 \in S_1$ if $\forall \varepsilon > 0, \exists \delta > 0,$ s.t. $\forall x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$.
And f is <u>uniformly continuous</u> on $E \subseteq S_1$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in E$ with $y \in B_\delta(x) \Rightarrow f(y) \in B_\varepsilon(x)$.

<u>Thm</u> (21.3)
A function $f: S_1 \to S_2$ is continuous on S_1 if and only if

$$\forall U \subseteq S_2 \text{ open } \Rightarrow f^{-1}(U) \subseteq S_1 \text{ is open as well.}$$

-Notes: 1) This theorem tells us that continuity can be defined through open sets.
2) Continuous functions do not have to map open sets to open sets (i.e. $f(x) = 1$).
-Moral: The inverse map of a continuous function preserves open sets.

<u>Thm</u> Let $f: S_1 \to S_2$ be continuous and $E \subseteq S_1$ be compact. Then
a) $f(E) \subseteq S_2$ is compact
b) f is uniformly continuous on E.

<u>Cor</u> Let $f: S_1 \to \mathbb{R}$ be continuous and $E \subseteq S_1$ be compact. Then f is bounded on E and obtains its maximum and minimum values on E.
-Note: This is the equivalent to theorems 18.1 and 19.2 when $S_1 = S_2 = \mathbb{R}$ and is the generalization of those theorems on an arbitrary metric space. Also, this tells us that compact sets are the generalization of closed sets on an arbitrary metric space