Math 125A
Vogler

# Properties of Continuous Functions

**Defn** $f$ is bounded if the range (i.e. $\{f(x) : x \in \text{dom } f\}$) is a bounded set (i.e. $\exists\, M$ such that $|f(x)| \le M \;\forall x$).

**Thm (18.1)**
Let $f$ be a continuous function on $[a,b]$. Then $f$ is bounded. Also, $f$ obtains it's maximum & minimum values on $[a,b]$ (i.e. $\exists\, \underline{x}, \overline{x} \in [a,b]$ such that $f(\underline{x}) \le f(x) \le f(\overline{x}) \quad \forall x \in [a,b]$).

- Notes: 1) This theorem is the reason we always found a solution to optimization problems in Math 21A. Back then we dealt with continuos functions on closed intervals (sometimes this was not necessary).
2) While this might be true on an open interval $(a,b)$, it isn't necessarily true (See HW#2 18.3)

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**Intermediate Value Thm (18.2)**
If $f$ is continuous on an interval $I$, then it satisfies
(1) When $a, b \in I$, $a < b$, and the number $m$ lies between $f(a)$ & $f(b)$, then $\exists$ at least one $x \in (a,b)$ with $f(x) = m$.

- Notes: 1) Sometimes (1) is referred to as the intermediate value property.
2) A consequence of this theorem is the output of $f$ on an interval $I$, denoted $f(I)$, is an interval or a single point. (See corollary 18.3)

- Moral: These two theorems tell us that a continuous function on a closed interval obtains not only a finite maximum and minimum output, but all the outputs between this maximum and minimum.

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**Thm** (18.4)
Let $f$ be a strictly increasing function on an interval $I$, and $J$ it's output interval (i.e. $f(I)=J$). Then the inverse function $f^{-1}$ is also a continuous strictly increasing function on $J$.

- Note: The same holds true for a strictly decreasing function.

**Thm** (18.6)
If $f$ is a one-to-one continuous function, $f$ must be either strictly increasing or strictly decreasing.

- Note: Combined with theorem 18.4, this implies all one-to-one continuous functions has a continuous inverse which is strictly increasing/decreasing.