Math 125A
Vogler

Taylor's Theorem

__Defn__ A function is __smooth__ at $x=a$ if $f^{(n)}(a)$ exists and is finite $\forall n \in \mathbb{N}$.

-Note: Another common term is f is infinitely differentiable.

__Defn__ Let f be defined on an open interval I with $\emptyset \in I$. If f is smooth at 0, then

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

is called the __Taylor Series for f about $\emptyset$.__

The __nth remainder__ $R_n(x)$ is defined by

$$R_n(x) := f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

-Notes: 1) For any x, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ iff $\lim_{n\to\infty} R_n(x) = 0$.

2) The remainder equation can be rewritten as

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + R_n(x), \text{ where}$$

$P_{n-1}(x) := \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k$ is refered to as the __Taylor Polynomial of degree $n-1$ about 0.__

3) In general, we can define the __Taylor Series for f about a__

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

and all the Theorems can be rewritten to apply to this form.

4) The most common use is to approximate a function by its tangent line (a.k.a. Taylor Polynomial of degree 1)

$$f(x) \approx f(0) + f'(0)\,x$$

-Moral: All smooth functions can be locally approximated by a polynomial of any degree.

Taylor's Error Formula (Thm 31.3)

Let f be defined on (a,b) with $0 \in (a,b)$, and assume $f^{(n)}$ exists on (a,b). Then $\forall x \in (a,b)$ where $x \neq 0$, $\exists c$ between 0 and x such that

$$R_n(x) = \frac{f^{(n)}(c)}{n!} x^n$$

-Note: Usually the formula to determine the max error of a Taylor Polynomial of degree $n-1$ (i.e. $|f(x) - P_{n-1}(x)|$).

Cor (31.4)

Let f be smooth on (a,b) with $0 \in (a,b)$. If $|f^{(n)}(x)| \leq M$ where M is a constant $\forall n \in \mathbb{N}, \forall x \in (a,b)$, then

$$\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in (a,b)$$

-Moral: All smooth fns with bounded derivatives equals its Taylor Series.

Taylor's Error Formula (Integral Form) (Thm 31.5)

Let f be defined on (a,b) with $0 \in (a,b)$, and assume $f^{(n)}$ exists and is continuous on (a,b). Then $\forall x \in (a,b)$,

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt.$$

-Note: This formula can be rewritten as (Cor. 31.6)

$$R_n(x) = \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y) \cdot x$$

for $x \in (a,b)$ where $x \neq 0$ and y between 0 and x.

Binomial Series Theorem

If $\alpha \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k.$$