Math 125A
Vogler

Uniform Continuity

Defn f is *uniformly continuous on* $S \subseteq \mathbb{R}$ if
(1) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in S$ w/ $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$.
f is called *uniformly continuous* if it's uniformly continuous on its domain.

-Notes: 1) Unlike continuity, uniform continuity cannot be defined at a point, only on a set.
2) Uniform continuity on S implies continuity on S, which is easily shown by letting $y = x_0$ and comparing to $\varepsilon\delta$-property.
3) The negation of (1) is
$\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x, y \in S$ with $|x-y| < \delta$ and $|f(x)-f(y)| \geq \varepsilon$.

Thm (19.2)
If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

-Notes: 1) With the continuity theorems (17.3 – 17.5) and list of known continuous fns, we can show most functions are uniformly continuous on any $[a,b]$ with this theorem.
2) The interval $[a,b]$ can be replaced with any closed and bounded set S, and the theorem still holds.

Moral: On closed and bounded sets continuity and uniform continuity are equivalent.

Thm (19.4)
If f is uniformly continuous on S and $\{s_n\}$ is a Cauchy sequence in S, then $\{f(s_n)\}$ is a Cauchy sequence.

- Note: This theorem gives us a way to prove/disprove uniform continuity using sequences and is the closest analog to the 'sequencial form of continuity'.
- Moral: Uniform continuity 'preserves' Cauchy sequences.

<u>Defn</u> The function $\hat{f}$ is an <u>extension of a function f</u> if domain$(f) \subseteq$ domain$(\hat{f})$ and $f(x) = \hat{f}(x)$ $\forall x \in$ domain(f).

<u>Thm</u> (19.5)
A function f is uniformly continuous on (a,b) if and only if f can be extended to a continuous function $\hat{f}$ on $[a,b]$.

- Note: Very useful theorem for showing:
a) Continuous functions with removable singularities (a.k.a. 'holes') are uniformly continuous once 'filled-in' correctly. (i.e. $f(x) = \frac{\sin x}{x}$)
b) Continuous functions with an undefined limits through oscillation are not uniformly continuous. (i.e. $f(x) = \sin(\frac{1}{x})$).

<u>Thm</u> (19.6)
Let f be continuous on interval I, and let I^o be all points in I without the endpoints. If f is differentiable on I^o along with f' bounded on I^o, then f is uniformly continuous on I.

- Note: I^o is sometimes refered to as the <u>interior of I</u> and is an open set (i.e. If $I = [a,b] \Rightarrow I^o = (a,b)$)
- Moral: A continuous function with bounded slope on its interior is uniformly continuous.