Math 125A
Vogler

Uniform Convergence

Defn A sequence of functions $\{f_n\}$ converges pointwise to a function f, written $\{f_n\} \to f$, on $S \subseteq \mathbb{R}$ if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in S.$$

Defn A sequence of functions $\{f_n\}$ converges uniformly to a function f, written $\{f_n\} \rightrightarrows f$, on $S \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \text{ such that } \forall x \in S, \forall n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

- Notes: 1) Uniform convergence is stronger than pointwise convergence (i.e. $\{f_n\} \rightrightarrows f \Rightarrow \{f_n\} \to f$).

2) If $\{f_n\} \rightrightarrows g_1$ and $\{f_n\} \to g_2 \Rightarrow g_1(x) = g_2(x)$ (i.e. the pointwise and uniform limits are unique). Hence, when trying to find the uniform limit one finds the pointwise limit as a candidate.

Thm (24.3)
Let $\{f_n\}$ be a sequence of functions defined on $S \subseteq \mathbb{R}$ where $\{f_n\} \rightrightarrows f$ on S. If f_n is continuous at x_0 for all n, then f is continuous at x_0.

- Note: This gives us a quick way of determining whether the limit of continuous functions is uniform by checking whether the limit function is continuous, assuming we can find the limit function.
- Moral: The uniform limit of continuous functions is continuous.

Prop (24.4)
$\{f_n\} \rightrightarrows f$ on $S \subseteq \mathbb{R}$ if and only if $\lim_{n\to\infty} [\sup\{|f(x) - f_n(x)| : x \in S\}] = 0$.

- Note: Another useful way of determining uniform convergence.
- Moral: A sequence of functions converging uniformly is equivalent to the biggest difference between a sequence element and the limit function tends to zero.

Defn
A sequence of functions $\{f_n\}$ defined on $S \subseteq \mathbb{R}$ is <u>uniformly</u> <u>Cauchy</u> if
$\forall \varepsilon > 0, \exists N$ such that $\forall x \in S, \forall m,n > N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$
-Notes: 1) A useful concept because it allows us to show uniform convergence without knowing the limit function.
2) It's easy to show uniform convergence $\Rightarrow$ uniform Cauchy.

Thm (25.4)
Let $\{f_n\}$ be a sequence of functions that are uniformly Cauchy on $S \subseteq \mathbb{R}$. Then there exists a function f such that $\{f_n\} \rightrightarrows f$ on S.

Defn
Let $\{g_k\}$ be a sequence of functions. The resulting series $\sum_{k=0}^{\infty} g_k(x)$ is called a <u>series</u> <u>of</u> <u>functions</u>, which makes sense if the sequence of partial sums $f_n(x) := \sum_{k=0}^{n} g_k(x)$ converges or diverges to $\pm\infty$ pointwise. If there exists an f such that $\{f_n\} \rightrightarrows f$ on $S \subseteq \mathbb{R}$, then we say the series is <u>uniformly</u> <u>convergent</u> on S.
-Note: If g_k is continuous for all k and $\sum g_k$ is uniformly convergent on S, then $f(x) := \sum g_k(x)$ is continuous on S by Thm 24.3.

Defn
The series of functions $\sum g_k$ satisfies the <u>Cauchy</u> <u>criterion</u> if
$\forall \varepsilon > 0, \exists N$ such that $\forall x \in S, \forall n \geq m > N \Rightarrow \left|\sum_{k=m}^{n} g_k(x)\right| < \varepsilon$.
-Notes: 1) This is equivalent to the seq. of partial sums being uniformly Cauchy.
2) The series satisfies the Cauchy criterion iff it's uniformly convergent.

<u>Weierstrass M-Test</u> Consider sequence $\{M_k\} \subseteq [0,\infty)$ with $\sum M_k < \infty$. If $|g_k(x)| \leq M_k \ \forall x \in S \subseteq \mathbb{R}$, then $\sum g_k$ converges uniformly on S.