21.6) Suppose $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are both continuous. Consider the function $g \circ f : S_1 \to S_3$. Let $U \subseteq S_3$ be open. Since g is continuous, $g^{-1}(U) \subseteq S_2$ is open by Theorem 21.3. Since f is continuous, $f^{-1}(g^{-1}(U)) \subseteq S_1$ is open by another application of Theorem 21.3. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ by the definition of composition and inverse. Thus, $(g \circ f)^{-1}(U) \subseteq S_1$ is open. Since U was arbitrary, this holds for all $U \subseteq S_3$. Therefore, $g \circ f$ is continuous by Theorem 21.3.

21.8) Suppose $f : S \to S^*$ is uniformly continuous and $\{s_n\} \subseteq S$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in S \text{ with } d(x, y) < \delta \Rightarrow d^*(f(x), f(y)) < \epsilon. \tag{1}$$

Since $\{s_n\}$ is Cauchy, for this δ

$$\exists N \text{ such that } \forall m, n > N \Rightarrow d(s_m, s_n) < \delta.$$

Then, for this N, we have

$$\forall m, n > N \Rightarrow d(s_m, s_n) < \delta \Rightarrow d^*(f(s_m), f(s_n)) < \epsilon,$$

by (1). Therefore, $\{f(s_n)\}$ is a Cauchy sequence as well.

21.10) (a) Let $f : (0, 1) \to [0, 1]$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} \\ 2x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} < x < 1 \end{cases}$$

(b) Let $g : (0, 1) \to \mathbb{R}$ be defined as $g(x) = \tan(\pi x - \frac{\pi}{2})$.

(c) Let $h : [01] \cup [2, 3] \to [0, 1]$ be defined as $h(x) = -\frac{1}{2}x^2 + \frac{3}{2}x$