23.6) (a) Suppose that $\sum a_n x^n$ has a finite radius of convergence $R < \infty$ and $a_n \geq 0$ $\forall n$. Also, assume the series converges at $x = R$, which by definition of convergence means $\sum a_n R^n < \infty$. We want to show $\sum a_n(-R)^n < \infty$. Notice that

$$\sum a_n(-R)^n = \sum (-1)^n a_n R^n$$

is an alternating series. Since $\sum a_n x^n$ converges and $a_n \geq 0$ $\forall n$, the sequence $\{a_n R^n\}$ must be positive, decreasing, and $\lim_{n\to\infty} a_n R^n = 0$. By the Alternating Series Test, the series $\sum a_n(-R)^n$ converges. Thus, the power series $\sum a_n x^n$ converges at $x = -R$.

23.8) Let the sequence of functions $\{f_n\}$ be defined by $f_n(x) = \frac{1}{n}\sin nx$ $\forall n$.

(a) Let $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{n}$$

Notice we have the following bound

$$-\frac{1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n} \quad \forall n$$

Clearly, the following limits are true

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0.$$

By the Squeeze Theorem (Exercise 8.5), we have

$$\lim_{n\to\infty} \frac{\sin nx}{n} = 0.$$

Since $x \in \mathbb{R}$ was arbitrary,

$$\lim_{n\to\infty} f_n(x) = 0 \ \forall x \in \mathbb{R}$$

(b) Consider the sequence of derivative functions $\{f_n'\}$ which are $f_n'(x) = \cos nx$ $\forall n$.

For $x = \pi$, we have $f_n'(\pi) = \cos n\pi$ $\forall n$. There's a difference in outputs when n is odd verses even. In particular,

$$f_n'(\pi) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

The sequence $\{f_n'(\pi)\}$ alternates between 1 and -1, so the limit of this sequence does not exist. Consequently, $\lim_{n\to\infty} f_n'(x)$ cannot exist.