24.2) For $x \in [0, \infty)$, let the sequence of functions $\{f_n\}$ be defined by $f_n(x) = \frac{x}{n} \forall n$. (a) For x = 0, we have

$$f_n(0) = 0 \ \forall n \Rightarrow \{f_n(0)\} \to 0.$$

For $x \in (0, \infty)$, we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x}{n} = 0$$

Thus, we choose to define $f(x) = 0 \ \forall x \in [0, \infty)$ so that $\{f_n\} \to f$ on $[0, \infty)$.

(b) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0,1]\} = \sup\{\frac{x}{n} : x \in [0,1]\} = \frac{1}{n}$$

Then

$$\lim_{n \to \infty} \sup\{|f(x) - f_n(x)| : x \in [0, 1]\} = \lim_{n \to \infty} \frac{1}{n} = 0$$

Thus, $\{f_n\} \Rightarrow f$ on [0,1] by Proposition in Remark 24.4.

(c) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0, \infty)\} = \sup\{\frac{x}{n} : x \in [0, \infty)\} = \infty.$$

Then

$$\lim_{n \to \infty} \sup\{|f(x) - f_n(x)| : x \in [0, \infty)\} = \infty.$$

Thus, $\{f_n\}$ does not uniformly converge to f on $[0,\infty)$ by Proposition in Remark 24.4.

24.6) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \left(x - \frac{1}{n}\right)^2$ be defined on $x \in [0, 1]$. (a) Let $x \in [0, 1]$. Then

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left(x - \frac{1}{n} \right)^2 = x^2$$

Thus, we choose to define $f(x) = x^2 \ \forall x \in [0,1]$ so that $\{f_n\} \to f$ on $x \in [0,1]$.

(b) Yes. Let $\epsilon > 0$ be given and $x \in [0, 1]$. Then we have

$$|f_n(x) - f(x)| = \left| (x - \frac{1}{n})^2 - x^2 \right| = \left| \frac{1}{n^2} - \frac{2x}{n} \right| \le \frac{1}{n^2} - \frac{2}{n}$$

from the Triangle Inequality and $x \in [0, 1]$. Since $n^2 > n \ \forall n \in \mathbb{N}$, we have the following bound

$$|f_n(x) - f(x)| \le \frac{1}{n^2} - \frac{2}{n} < \frac{3}{n}.$$

 So

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \frac{3}{n} < \epsilon \Leftrightarrow n > \frac{3}{\epsilon}$$

Choose $N = \frac{3}{\epsilon}$. Thus,

$$\forall n > N \Rightarrow |f_n(x) - f(x)| < \epsilon$$

Since $x \in [0, 1]$ was arbitrary, it holds for all $x \in [0, 1]$. Therefore, $\{f_n\} \Rightarrow f$ on [0, 1] by definition.

24.8) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \sum_{k=0}^n x^k$ be defined on $x \in [0, 1]$. (a) For x = 1, $f_n(1) = n \forall n$. Clearly,

$$\lim_{n \to \infty} f_n(1) = \infty,$$

so the limit does not exist. Therefore, the sequence $\{f_n\}$ does not converge pointwise on [0, 1].

(b) No. Since $\{f_n\}$ does not converge pointwise on [0,1] (see part a), $\{f_n\}$ cannot converge uniformly [0,1].

24.10) Suppose $\{f_n\} \Rightarrow f$ and $\{g_n\} \Rightarrow g$ on a set S. Consider the sequence of functions $\{f_n + g_n\}$ on S. Let $\epsilon > 0$ be given and $x \in S$. Notice

$$|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)| \le |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

by the Triangle Inequality. Consider the number $\frac{\epsilon}{2} > 0$. Since $\{f_n\} \rightrightarrows f$, there exists N_1 such that

$$\forall n > N_1 \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}.$$
(1)

Also, since $\{g_n\} \rightrightarrows g$, there exists N_2 such that

$$\forall n > N_2 \Rightarrow |g_n(x) - g(x)| < \frac{\epsilon}{2}.$$
(2)

Choose $N = \max\{N_1, N_2\}$. Then

$$\forall n > N \Rightarrow |(f_n + g_n)(x) - (f + g)(x)| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by (1) and (2). Since $x \in S$ was arbitrary, it holds for all $x \in S$. Therefore, $\{f_n + g_n\} \rightrightarrows f + g$