25.2) Let the sequence of functions $\{f_n\}$ be defined by $f_n(x) = \dfrac{x^n}{n}$ on $[-1, 1]$. First we find the pointwise limit f. Let $x \in [-1, 1]$. Then we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{n}.$$

Since $|x| \leq 1$, the following inequality holds

$$-\frac{1}{n} \leq \frac{x^n}{n} \leq \frac{1}{n} \quad \forall n.$$

Clearly, the following limits are true

$$\lim_{x\to\infty} \frac{1}{n} = \lim_{x\to\infty} -\frac{1}{n} = 0.$$

Then,

$$\lim_{n\to\infty} \frac{x^n}{n} = 0$$

by the Squeeze Theorem (Exercise 8.5). Thus, we choose to define $f(x) = 0 \ \forall x \in [-1, 1]$ so that $\{f_n\} \to f$ on $x \in [-1, 1]$.

Now we show the uniform convergence. Let $\epsilon > 0$ be given and $x \in [-1, 1]$. We have

$$|f_n(x) - f(x)| = \left|\frac{x^n}{n} - 0\right| = \left|\frac{x^n}{n}\right| \leq \frac{1}{n}$$

So

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$$

Choose $N = \dfrac{1}{\epsilon}$. Thus,

$$\forall n > N \Rightarrow |f_n(x) - f(x)| < \epsilon$$

Since $x \in [-1, 1]$ was arbitrary, it holds for all $x \in [-1, 1]$. Therefore, $\{f_n\} \rightrightarrows f$ on $[-1, 1]$ by definition.

25.4) Let the sequence of functions $\{f_n\}$ on $S \subseteq \mathbb{R}$. Suppose $\{f_n\} \rightrightarrows f$ on S. Let $\epsilon > 0$ be given. Note

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \ \forall x \in S$$

by the Triangle Inequality.

Consider the number $\dfrac{\epsilon}{2} > 0$. Since $\{f_n\} \rightrightarrows f$, there exists N such that

$$\forall n > N \quad \forall x \in S \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}. \tag{1}$$

Moreover, we have

$$\forall m > N \quad \forall x \in S \Rightarrow |f_m(x) - f(x)| < \frac{\epsilon}{2}. \tag{2}$$

Thus,

$$\forall m, n > N \quad \forall x \in S \Rightarrow |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by (1) and (2). Therefore, $\{f_n\}$ is uniformly Cauchy on S by definition.

25.6) (a) Suppose $\sum |a_k| < \infty$ (i.e. a convergent series of numbers). Here we have a sequence $\{|a_k|\}$ of nonnegative numbers with $\sum |a_k| < \infty$. Consider the power series $\sum a_k x^k$ on $[-1, 1]$. Since $|x| \leq 1$, we have

$$|a_k x^k| = |a_k||x^k| \leq |a_k| \ \forall k \text{ and } \forall x \in [-1, 1].$$

Thus, the power series $\sum a_k x^k$ converges uniformly on $[-1, 1]$ by the Weierstrass M-Test. Clearly, a power series is a series of continuous functions (since they are just polynomials). Therefore, $\sum a_k x^k$ converges uniformly to a continuous functions by Theorem 25.5.

(b) Yes. Since $a_k = \dfrac{1}{k^2} > 0 \ \forall k$, and $\sum \dfrac{1}{k^2}$ is a convergent p-series, the power series $\sum \dfrac{1}{k^2} x^k$ converges uniformly to a continuous function on $[-1, 1]$ by the assertion proved in part (a).

25.12) Suppose $\sum g_k$ is a series of continuous functions g_k on $[a, b]$ that converges uniformly to g on $[a, b]$. Define the corresponding sequence of partial sums $\{f_n\}$ defined by $f_n(x) = \sum_{k=1}^{n} g_k(x)$ for all n and $x \in [a, b]$. Notice for all n that f_n is continuous (since addition preserves continuity), and we have $\{f_n\} \rightrightarrows g$ on $[a, b]$ by definition of uniform convergence on a series of functions. From Theorem 25.2, we have

$$\int_a^b g(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx = \lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} g_k(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b g_k(x)\ dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\ dx$$

since the integral 'distributes' over addition (Theorem 35.8), proving the claim.

25.14) Suppose $\sum g_k$ is a series of functions that converges uniformly to g on S, and h is a bounded function on S. h is bounded on S means there exists an $M \in \mathbb{R}$ such that $|h(x)| \leq M \ \forall x \in S$. Notice

$$|\sum_{k=1}^{n} h(x)g_k(x) - h(x)g(x)| = |h(x)||\sum_{k=1}^{n} g_k(x) - g(x)| \leq M|\sum_{k=1}^{n} g_k(x) - g(x)|$$

Let $\epsilon > 0$ be given, and consider the value $\dfrac{\epsilon}{M} > 0$. Since the series of functions converges uniformly to g on S, there exists an N such that

$$\forall n > N \ \ \forall x \in S \Rightarrow |\sum_{k=1}^{n} g_k(x) - g(x)| < \frac{\epsilon}{M}.$$

For this N, we have

$$\forall n > N \ \ \forall x \in S \Rightarrow |\sum_{k=1}^{n} h(x)g_k(x) - h(x)g(x)| \leq M|\sum_{k=1}^{n} g_k(x) - g(x)| < M\frac{\epsilon}{M} = \epsilon.$$

Therefore, the series of functions $\sum hg_k$ converges uniformly to hg on S by definition.