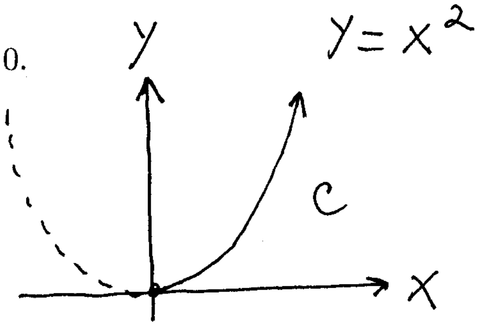
3.) Let path C be determined by $\vec{r}(t) = (t)\,\vec{i} + (t^2)\,\vec{j}$ for $t \geq 0$.

a.) Plot C in the xy-plane.

$$\begin{cases} x = t \\ y = t^2 \end{cases} \text{ for } t \geq 0 \rightarrow y = x^2$$



b.) Find the velocity vector $\vec{v}(t)$, the acceleration vector $\vec{a}(t)$, the unit tangent vector $\vec{T}(t)$, and the principal unit normal vector $N(t)$.

$$\xrightarrow{D} \underline{\underline{\vec{v}(t) = (1)\vec{i} + (2t)\vec{j}}} \xrightarrow{D} \underline{\underline{\vec{a}(t) = (2)\vec{j}}} \rightarrow$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{\sqrt{1+4t^2}} = \underline{\underline{(1+4t^2)^{-1/2}\,\vec{i} + \frac{2t}{(1+4t^2)^{1/2}}\,\vec{j}}}$$

$$\xrightarrow{D} \vec{T}'(t) = -\frac{1}{2}(1+4t^2)^{-3/2}(8t)\,\vec{i}$$

$$+ \frac{(1+4t^2)^{1/2}(2) - 2t\cdot\frac{1}{2}(1+4t^2)^{-1/2}\cdot 8t}{(1+4t^2)}\,\vec{j}$$

$$= \frac{-4t}{(1+4t^2)^{3/2}}\,\vec{i} + \left[\frac{2(1+4t^2)^{1/2}}{1} - \frac{8t^2}{(1+4t^2)^{1/2}}\right]\cdot\frac{1}{1+4t^2}\,\vec{j}$$

$$= \frac{-4t}{(1+4t^2)^{3/2}}\,\vec{i} + \frac{2(1+4t^2) - 8t^2}{(1+4t^2)^{3/2}}\,\vec{j} = \frac{-4t}{(1+4t^2)^{3/2}}\,\vec{i} + \frac{2}{(1+4t^2)^{3/2}}\,\vec{j}, \text{ so}$$

$$|\vec{T}'(t)| = \sqrt{\left(\frac{-4t}{(1+4t^2)^{3/2}}\right)^2 + \left(\frac{2}{(1+4t^2)^{3/2}}\right)^2} = \sqrt{\frac{4(1+4t^2)}{(1+4t^2)^3}} = \frac{2}{1+4t^2}$$

$$\rightarrow \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{-2t}{\sqrt{1+4t^2}}\,\vec{i} + \frac{1}{\sqrt{1+4t^2}}\,\vec{j}$$

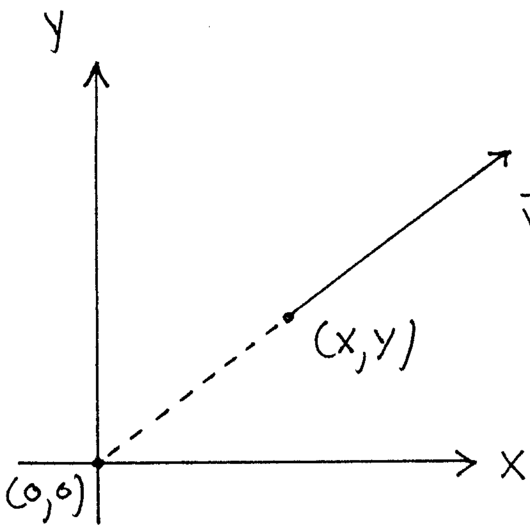
c.) Find the speed and acceleration of motion when $t = 1$.

speed $\frac{ds}{dt} = |\vec{v}(1)| = \sqrt{1+4(1)^2} = \sqrt{5}$, acceleration is

$$a(1) = \frac{\vec{v}(1)\cdot\vec{a}(1)}{|\vec{v}(1)|} = \frac{(\vec{i}+2\vec{j})\cdot(2\vec{j})}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$

6.) Give a formula $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ for a vector field in the plane that has the following two properties :

i.) It points away from the origin.
ii.) Its magnitude is equal to the cube of the distance from (x,y) to the origin.



$$\vec{v} = (x-0)\vec{i} + (y-0)\vec{j} = (x)\vec{i} + (y)\vec{j}$$

$$\rightarrow \vec{u} = \frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j} \;;$$

unit vector

magnitude is $\left(\sqrt{x^2+y^2}\right)^3 = (x^2+y^2)^{3/2}$;

then

$$\vec{F}(x,y) = (x^2+y^2)^{3/2} \cdot \vec{u}$$

$$= (x^2+y^2)x \cdot \vec{i} + (x^2+y^2)y \cdot \vec{j}$$

6.) Let path C be given by $\vec{r}(t) = (e^t - t)\,\vec{i} + (4e^{(1/2)t})\,\vec{j} + 3\,\vec{k}$ for $0 \le t \le 2$. Compute the length of C.

$$\text{Arc} = \int_0^2 \sqrt{(e^t-1)^2 + (2e^{\frac{1}{2}t})^2}\,dt$$

$$= \int_0^2 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\,dt$$

$$= \int_0^2 \sqrt{(e^t+1)^2}\,dt = \int_0^2 (e^t+1)\,dt = (e^t + t)\Big|_0^2$$

$$= (e^2 + 2) - (e^0 + 0) = e^2 + 1$$

7.) Let path C be given by $\vec{r}(t) = (5\cos t)\,\vec{i} + (5\sin t)\,\vec{j} + (12t)\,\vec{k}$ for $t \geq 0$. Write time t as a function of arc length s.

$$s(t) = \int_0^t \sqrt{(-5\sin\tau)^2 + (5\cos\tau)^2 + (12)^2}\, d\tau$$

$$= \int_0^t \sqrt{25\sin^2\tau + 25\cos^2\tau + 144}\, d\tau$$

$$= \int_0^t \sqrt{25(\underbrace{\sin^2\tau + \cos^2\tau}_{1}) + 144}\, d\tau = \int_0^t 13\, d\tau$$

$$= 13\tau \Big|_0^t = 13t \rightarrow s = 13t \rightarrow$$

$$t = \frac{1}{13} s$$

8.) Evaluate the following line integrals

b.) $\int_C xz\ ds$, where C : line segment from $(3, 0, -1)$ to $(2, 2, 1)$

direction vector for line segment is

$$\vec{w} = (2-3)\,\vec{i} + (2-0)\,\vec{j} + (1-(-1))\,\vec{k} = (-1)\,\vec{i} + (2)\,\vec{j} + (2)\,\vec{k}$$

so

$$C: \begin{cases} x = 3 + (-1)t \\ y = 0 + (2)t \\ z = -1 + (2)t \end{cases} \text{ for } 0 \leq t \leq 1$$

$$\vec{r}(t) = (3-t)\,\vec{i} + (2t)\,\vec{j} + (-1+2t)\,\vec{k} \quad D$$

$$\vec{v}(t) = (-1)\,\vec{i} + (2)\,\vec{j} + (2)\,\vec{k} \text{ so}$$

$$|\vec{v}(t)| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3,$$

then

$$\int_C xz\, ds = \int_0^1 (3-t)(-1+2t)\cdot \frac{ds}{dt}\, dt = \int_0^1 (-2t^2 + 7t - 3)(3)\, dt$$

$$= 3\left(\frac{-2}{3}t^3 + \frac{7}{2}t^2 - 3t\right)\Big|_0^1 = 3\left(\frac{-2}{3} + \frac{7}{2} - 3\right)$$

$$= -2 + \frac{21}{2} - 9 = \frac{-22}{2} + \frac{21}{2} = \frac{-1}{2}$$

1.) Find the Flux for $\vec{F}(x,y) = (xy)\,\vec{\imath} + (x^2)\,\vec{\jmath}$ across the loop C : the ellipse $\left(\frac{x}{16}\right)^2 + \left(\frac{y}{9}\right)^2 = 1$.

$$C: \begin{cases} x = 16\cos t \\ y = 9\sin t \end{cases} \quad \text{for } 0 \le t \le 2\pi \quad \longrightarrow$$

$$\frac{dx}{dt} = -16\sin t, \quad \frac{dy}{dt} = 9\cos t \quad ;$$

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} \left((xy)\frac{dy}{dt} - (x^2)\frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} \left[(16\cos t)(9\sin t)(9\cos t) - (256\cos^2 t)(-16\sin t)\right] dt$$

$$= \int_0^{2\pi} 5392\cos^2 t\sin t\,dt$$

$$= -\frac{5392}{3}\cos^3 t \Big|_0^{2\pi}$$

$$= -\frac{5392}{3}\cos^3 2\pi - -\frac{5392}{3}\cos^3 0$$

$$= -\frac{5392}{3}(1)^3 + \frac{5392}{3}(1)^3$$

$$= 0$$

2.) Show that the following vector field is conservative. Then find a scalar function $f(x,y,z)$ satisfying $\vec{F} = \vec{\nabla} f(x,y,z)$.

$$\vec{F}(x,y,z) = \underbrace{(y\cos z - yze^{x})}_{M}\,\vec{i} + \underbrace{(x\cos z - ze^{x})}_{N}\,\vec{j} + \underbrace{(-xy\sin z - ye^{x} + 1)}_{P}\vec{k}$$

TEST: $M_y = \cos z - ze^x = N_x$,
$N_z = -x\sin z - e^x = P_y$,
$P_x = -y\sin z - ye^x = M_z$;

$f_x = y\cos z - yze^x \xrightarrow{S_x} f = xy\cos z - yze^x + g(y,z)$

$\xrightarrow{D_y} f_y = x\cos z - ze^x + g_y(y,z) = x\cos z - ze^x$

$\longrightarrow \quad g_y(y,z) = 0 \xrightarrow{S_y} g(y,z) = h(z) \longrightarrow$

$f = xy\cos z - yze^x + h(z) \xrightarrow{D_z}$

$f_z = -xy\sin z - ye^x + h'(z)$

$= -xy\sin z - ye^x + 1 \longrightarrow h'(z) = 1 \longrightarrow$

$h(z) = z + \cancel{c}^{\,0}$ so

$f = xy\cos z - yze^x + z$

3.) Show that $\int_{(0,0)}^{(1,2)} (y^2 + 2xy)dx + (x^2 + 2xy)dy$ is path independent. Then calculate value of line integral.

Let $\vec{F}(x,y) = (\underbrace{y^2 + 2xy}_{M})\vec{i} + (\underbrace{x^2 + 2xy}_{N})\vec{j}$, then

$$\int_{(0,0)}^{(1,2)} (y^2 + 2xy)\,dx + (x^2 + 2xy)\,dy = \int_{(0,0)}^{(1,2)} \vec{F}\cdot\vec{T}\,ds,$$

a work integral; and $M_y = 2y + 2x = N_x$ so $\vec{F}$ is conservative iff $\vec{F} = \vec{\nabla} f$;

find f:

$$f_x = y^2 + 2xy \xrightarrow{\int_x} f = xy^2 + x^2y + g(y)$$

$$\xrightarrow{D_y} f_y = 2xy + x^2 + g'(y) = x^2 + 2xy \rightarrow g'(y) = 0$$

$\rightarrow g(y) = C = 0$, so $\underline{f = xy^2 + x^2y}$; then

$$\int_{(0,0)}^{(1,2)} \vec{F}\cdot\vec{T}\,ds = \int_{(0,0)}^{(1,2)} \vec{\nabla} f\cdot\vec{T}\,ds$$

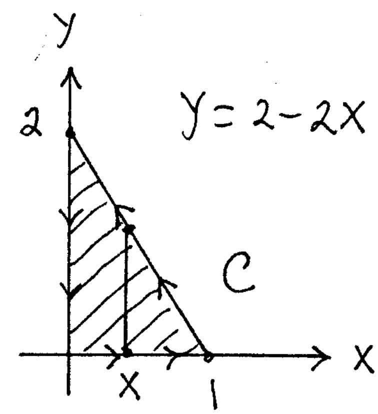
$$= f(x,y)\Big|_{(0,0)}^{(1,2)} = f(1,2) - f(0,0)$$

fundamental theorem for line integrals

$$= 6 - 0$$

$$= 6$$

5.) Use Green's Theorem to find the Circulation of $\vec{F}(x,y) = (x^2+y^2)\,\vec{\imath}+(-2xy)\,\vec{\jmath}$ around the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$.



$$\text{Circ} = \oint_C \vec{F}\cdot\vec{T}\,ds$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R \left((-2y)-(2y)\right) dA$$

$$= \int_0^1\int_0^{2-2x} -4y\,dy\,dx$$

$$= \int_0^1 \left(-2y^2 \Big|_{y=0}^{y=2-2x}\right) dx$$

$$= \int_0^1 \left[-2(2-2x)^2 - -2(0)^2\right] dx$$

$$= \int_0^1 -2(4x^2-8x+4)\,dx$$

$$= \int_0^1 (-8x^2+16x-8)\,dx$$

$$= \left(-\frac{8}{3}x^3+8x^2-8x\right)\Big|_0^1$$

$$= -\frac{8}{3}+8-8$$

$$= -\frac{8}{3}$$

The following EXTRA CREDIT PROBLEM is OPTIONAL.

1.) In class we showed that the integral for the Flux of the vector field $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ on loop $C : \vec{r}(t) = (f(t))\,\vec{i} + (g(t))\,\vec{j}$ for $a \le t \le b$ is

$$\int_C \vec{F}\cdot\vec{n}\;ds = \int_C M dy - N dx \;.$$

Show that Flux can also be calculated using

$$\int_C \vec{F}\cdot\vec{n}\;ds = -\int_C \vec{F}\cdot\vec{a}(t)\;\frac{|\vec{v}(t)|}{|\vec{a}(t)|}\;dt\;,$$

where $\vec{v}(t)$ and $\vec{a}(t)$ are the velocity vector and acceleration vector for $\vec{r}(t)$, resp.

assume that C is convex, so that $\vec{n} = -\vec{N}$;
then

$$\int_C \vec{F}\cdot\vec{n}\,ds = -\int_C \vec{F}\cdot\vec{N}\,ds = \int_C \vec{F}\cdot\frac{\vec{T}'(t)}{|\vec{T}'(t)|}\,ds$$

$$= -\int_C \vec{F}\cdot\frac{\frac{d}{dt}\left(\frac{\vec{v}(t)}{|\vec{v}(t)|}\right)}{\left|\frac{d}{dt}\left(\frac{\vec{v}(t)}{|\vec{v}(t)|}\right)\right|}\,ds$$

$$= -\int_C \vec{F}\cdot\frac{\frac{\vec{v}'(t)}{|\vec{v}(t)|}}{\left|\frac{\vec{v}'(t)}{|\vec{v}(t)|}\right|}\;\frac{ds}{dt}\,dt$$

$$= -\int_C \vec{F}\cdot\frac{\vec{a}(t)}{|\cancel{\vec{v}(t)}|}\cdot\frac{|\cancel{\vec{v}(t)}|}{|\vec{a}(t)|}\;|\vec{v}(t)|\,dt$$

$$= -\int_C \vec{F}\cdot\vec{a}(t)\cdot\frac{|\vec{v}(t)|}{|\vec{a}(t)|}\,dt\;.$$