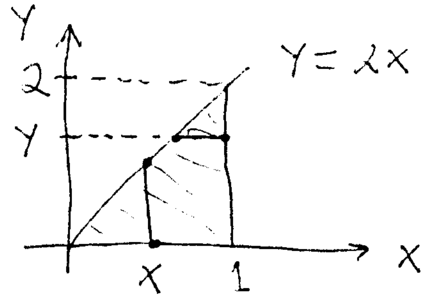
1.) (12 pts.) Evaluate $\displaystyle\int_0^2 \int_{(1/2)y}^1 e^{x^2}\, dx\, dy$.

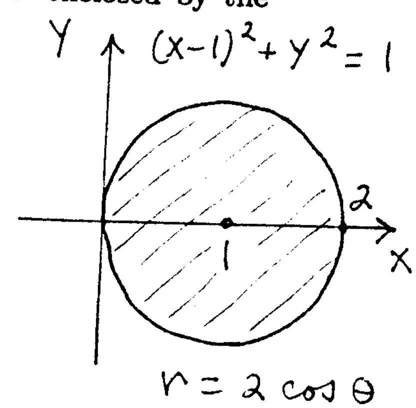


$$= \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left(e^{x^2} y \Big|_{y=0}^{y=2x} \right) dx = \int_0^1 2x e^{x^2} dx$$

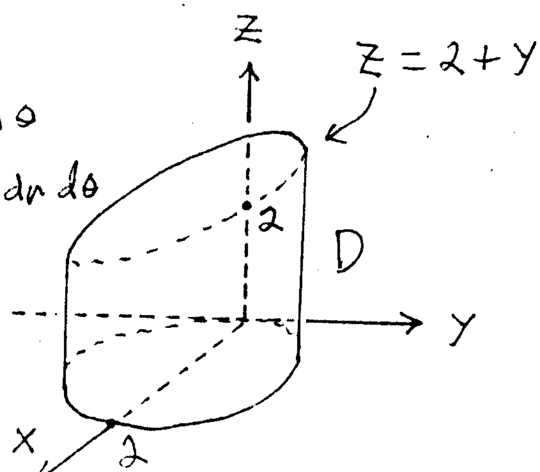
$$= e^{x^2} \Big|_0^1 = e^1 - e^0 = e - 1$$

2.) (12 pts.) Use a triple integral to find the volume of the region D enclosed by the cylinder $(x-1)^2 + y^2 = 1$, the plane $z = 0$, and the plane $z = 2 + y$.



$$\text{Vol } D = \iiint_D 1\, dV$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_0^{2 + r\sin\theta} r\, dz\, dr\, d\theta$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \left(rz \Big|_{z=0}^{z=2+r\sin\theta} \right) dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2r + r^2 \sin\theta)\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(r^2 + \tfrac{1}{3} r^3 \sin\theta \right) \Big|_{r=0}^{r=2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4\cos^2\theta + \tfrac{8}{3}\cos^3\theta \sin\theta \right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left(4 \cdot \tfrac{1}{2}(1 + \cos 2\theta) + \tfrac{8}{3}\cos^3\theta \sin\theta \right) d\theta$$

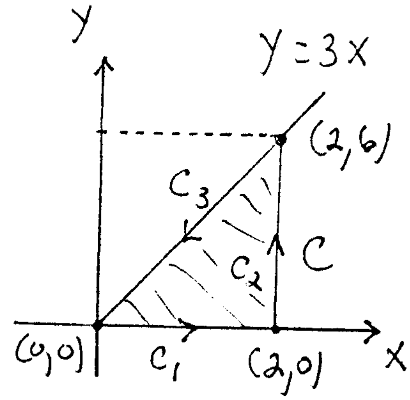
$$= \left(2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + \tfrac{8}{3} \cdot \tfrac{-1}{4}\cos^4\theta \right) \Big|_{-\pi/2}^{\pi/2}$$

$$= \left(2\left(\tfrac{\pi}{2} + \tfrac{1}{2}\underbrace{\sin\pi}_{0}\right) - \tfrac{2}{3}\underbrace{\cos^4\tfrac{\pi}{2}}_{0} \right) - \left(2\left(-\tfrac{\pi}{2} + \tfrac{1}{2}\underbrace{\sin(-\pi)}_{0}\right) - \tfrac{2}{3}\underbrace{\cos^4\left(-\tfrac{\pi}{2}\right)}_{0} \right) = 2\pi$$

3.) (7 pts. each) Let loop C be the triangle with vertices $(0,0)$, $(2,0)$, and $(2,6)$. Evaluate the line integral $\oint \underbrace{xy}_{M}\, dx + \underbrace{(x-y)}_{N}\, dy$ two ways :

a.) directly as a line integral. $\searrow$ $\vec{F}(x,y) = (xy)\vec{i} + (x-y)\vec{j}$

b.) using one of Green's Theorems.



a.) $\oint_C xy\, dx + (x-y)\, dy$

$= \int_{C_1} \cancel{(x\cdot 0)}^{\,0} dx + \cancel{(x-0)}^{\,0}(0) + \int_{C_2} (2y)(0) + (2-y)\,dy$

$+ \int_{C_3} x\cdot(3x)\, dx + (x-3x)\, dy$

$= \int_0^6 (2-y)\,dy + \int_2^0 3x^2\, dx + (-2x)\overset{3}{\frac{dy}{dx}}\, dx$

$= (2y - \frac{1}{2}y^2)\Big|_0^6 + \int_2^0 (3x^2 - 6x)\, dx$

$= (12-18) + (x^3 - 3x^2)\Big|_2^0 = -6 + (0 - (8-12)) = -2$;

b.) $\oint_C xy\, dx + (x-y)\, dy = \oint_C \vec{F}\cdot\vec{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= \iint_R (1-x)\, dA = \int_0^2 \int_0^{3x} (1-x)\, dy\, dx$

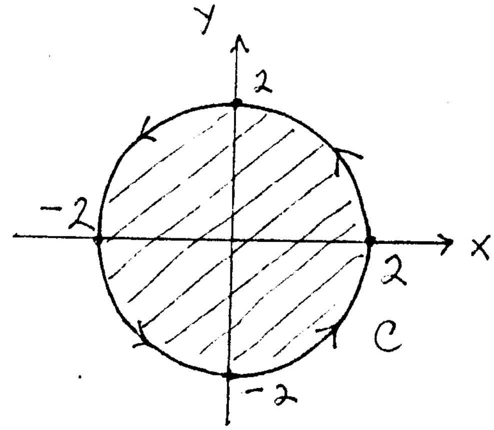
$= \int_0^2 (1-x)y\Big|_{y=0}^{y=3x} dx = \int_0^2 (1-x)3x\, dx$

$= \int_0^2 (3x - 3x^2)\, dx = (\frac{3}{2}x^2 - x^3)\Big|_0^2$

$= 6 - 8 = -2$

4.) (10 pts.) Let loop C be the circle $x^2 + y^2 = 4$. Use one of Green's Theorems to find the Flux of $\vec{F}(x,y) = (5x)\vec{i} + (-3y)\vec{j}$ across C.

$M \quad N$



$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds$$

$$= \oint_C M\,dy - N\,dx$$

$$= \iint_R \operatorname{div} \vec{F}\, dA$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

$$= \iint_R (5 + (-3))\, dA$$

$$= \iint_R 2\, dA$$

$$= 2 \iint_R 1\, dA$$

$$= 2\,(\text{area } R)$$

$$= 2 \cdot \pi (2)^2$$

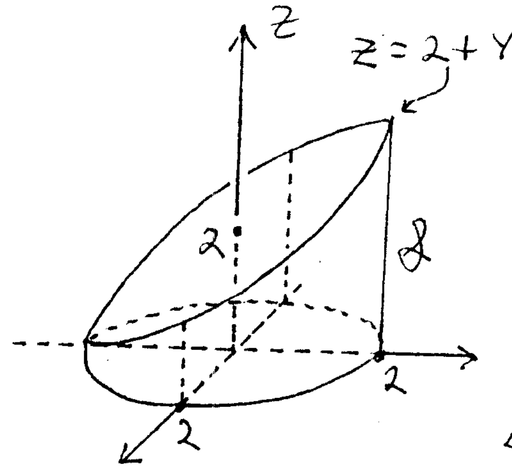
$$= 8\pi$$

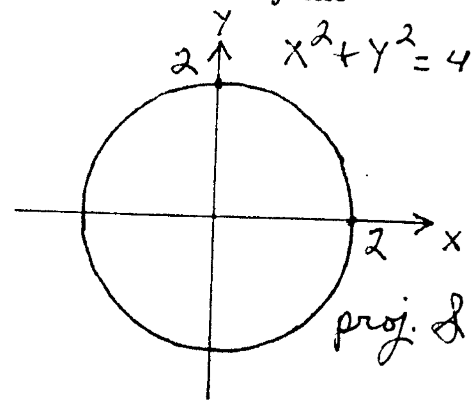
5.) (6 pts. each) Let surface S be that portion of the cylinder $x^2 + y^2 = 4$ cut by the planes $z = 0$ and $z = 2 + y$.

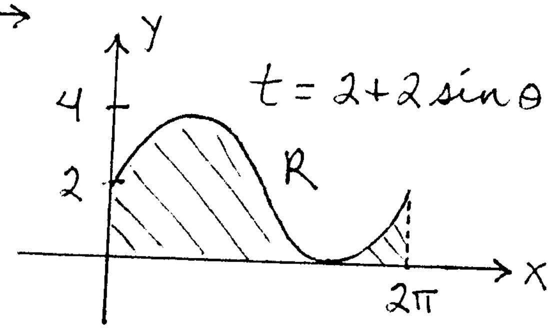
a.) Sketch surface S and parametrize it.

$$\mathcal{S}: \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = t \end{cases}$$

$$0 \le \theta \le 2\pi$$
$$0 \le t \le 2 + 2\sin\theta$$







b.) Use a surface integral to find the area of S.

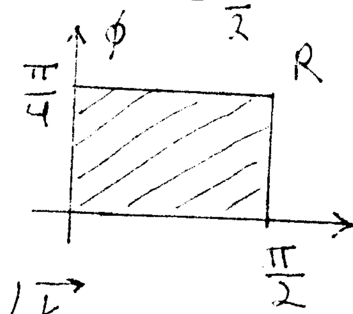
$$\vec{r}_\theta = (-2\sin\theta)\,\vec{i} + (2\cos\theta)\,\vec{j} + (0)\,\vec{k}$$
$$\vec{r}_t = (0)\,\vec{i} + (0)\,\vec{j} + (1)\,\vec{k}$$
$$\vec{r}_\theta \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= (2\cos\theta - 0)\,\vec{i} - (-2\sin\theta - 0)\,\vec{j} + (0 - 0)\,\vec{k}$$
$$= (2\cos\theta)\,\vec{i} + (2\sin\theta)\,\vec{j};$$

$$|\vec{r}_\theta \times \vec{r}_t| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2} = \sqrt{4(\underbrace{\cos^2\theta + \sin^2\theta}_{1})} = 2;$$

$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_t|\, dA = \int_0^{2\pi}\int_0^{2+2\sin\theta} 2\, dt\, d\theta$$
$$= \int_0^{2\pi} \left(2t \Big|_{t=0}^{t=2+2\sin\theta} \right) d\theta = \int_0^{2\pi} (4 + 4\sin\theta)\, d\theta$$
$$= (4\theta - 4\cos\theta)\Big|_0^{2\pi} = (8\pi - 4\overset{1}{\cancel{\cos 2\pi}}) - (0 - 4\overset{1}{\cancel{\cos 0}})$$
$$= 8\pi$$

6.) (12 pts.) Evaluate the surface integral $\displaystyle\int\!\!\int_S (xz)\ dS$, where S is given parametrically by $\vec{r}(\phi,\theta) = (\sin\phi\cos\theta)\ \vec{i} + (\sin\phi\sin\theta)\ \vec{j} + (\cos\phi)\ \vec{k}$ for $0 \le \phi \le \pi/4$ and $0 \le \theta \le \frac{\pi}{2}$.

$$S: \begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \quad \text{for } 0 \le \phi \le \frac{\pi}{4},\quad 0 \le \theta \le \frac{\pi}{2};$$



$$\vec{r}_\phi = (\cos\phi\cos\theta)\vec{i} + (\cos\phi\sin\theta)\vec{j} + (-\sin\phi)\vec{k},$$

$$\vec{r}_\theta = (-\sin\phi\sin\theta)\vec{i} + (\sin\phi\cos\theta)\vec{j} + (0)\vec{k},$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (0 - -\sin^2\phi\cos\theta)\vec{i} - (0 - \sin^2\phi\sin\theta)\vec{j} + (\sin\phi\cos\phi\cos^2\theta - -\sin\phi\cos\phi\sin^2\theta)\vec{k}$$

$$= (\sin^2\phi\cos\theta)\vec{i} + (\sin^2\phi\sin\theta) + \sin\phi\cos\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{1})\vec{k}, \text{ so}$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{(\sin^2\phi\cos\theta)^2 + (\sin^2\phi\sin\theta)^2 + (\sin\phi\cos\phi)^2}$$

$$= \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{\sin^4\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi(\underbrace{\sin^2\phi+\cos^2\phi}_{1})}$$

$$= \sin\phi; \text{ then}$$

$$\iint_S (xz)\,dS = \iint_R (xz)\cdot|\vec{r}_\phi\times\vec{r}_\theta|\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} (\sin\phi\cos\theta)(\cos\phi)(\sin\phi)\,d\phi\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \sin^2\phi\cos\phi\cos\theta\,d\phi\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{3}\sin^3\phi\cos\theta \Big|_{\phi=0}^{\phi=\frac{\pi}{4}}\right) d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{3}\sin^3\frac{\pi}{4}\cos\theta - \frac{1}{3}\sin^3 0\cdot\cos\theta\right) d\theta$$

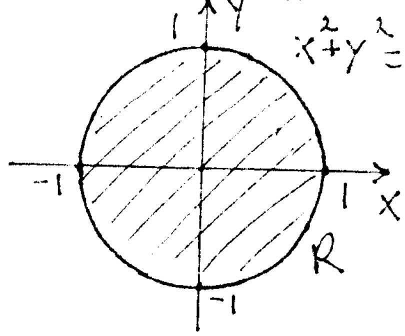
$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 \cos\theta\,d\theta = \frac{1}{3}\cdot\frac{2\sqrt{2}}{8}\sin\frac{\pi}{2} - \frac{1}{3}\cdot\frac{2\sqrt{2}}{8}\sin 0 = \frac{\sqrt{2}}{12}$$

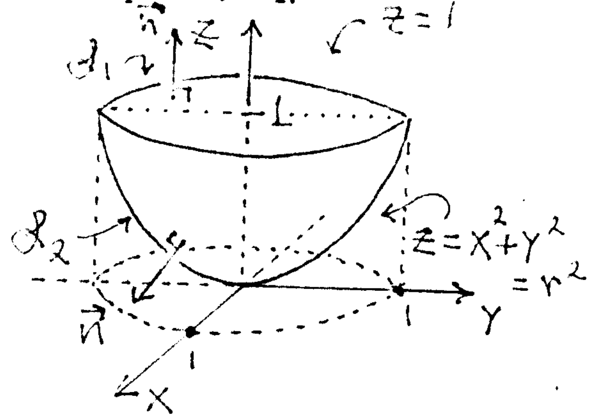
7.) (14 pts.) Verify the Divergence Theorem for $\vec{F}(x,y,z) = (y)\,\vec{i} + (-x)\,\vec{j} + (-xz)\,\vec{k}$, where the solid D is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

solid: D

surface: $S = S_1\,(\text{top}) \cup S_2\,(\text{side})$

$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$
$$= (0) + (0) + (-x)$$
$$= -x$$





Divergence Theo.: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \text{div}\,\vec{F}\,dV$;

I.) $$\iiint_D \text{div}\,\vec{F}\,dV = \iiint_D -x\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 -r\cos\theta\cdot r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 -r^2\cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(-r^2\cos\theta\cdot z\Big|_{z=r^2}^{z=1}\right)dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 \left(-r^2\cos\theta - {}^{-}r^2\cos\theta\cdot r^2\right)dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(-r^2\cos\theta + r^4\cos\theta\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{-r^3}{3}\cos\theta + \frac{r^5}{5}\cos\theta\right)\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\left(\frac{-1}{3}+\frac{1}{5}\right)\cos\theta\,d\theta$$
$$= \frac{-2}{15}\sin\theta\Big|_0^{2\pi} = \frac{-2}{15}\cancel{\sin 2\pi}^{\,0} - \frac{-2}{15}\cancel{\sin 0}^{\,0} = 0;$$

II.) $$\iint_{S_1}\vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{k}\,dS = \iint_{S_1} -xz\,dS = \iint_{S_1} -x(1)\,dS$$
$$= \int_0^{2\pi}\int_0^1 -r\cos\theta\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 -r^2\cos\theta\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(-\frac{1}{3}r^3\cos\theta\right)\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{-1}{3}\cos\theta\,d\theta$$
$$= \frac{-1}{3}\sin\theta\Big|_0^{2\pi} = \frac{-1}{3}\cancel{\sin 2\pi}^{\,0} - \frac{-1}{3}\cancel{\sin 0}^{\,0} = 0;$$

$\displaystyle\iint_{S_2}\vec{F}\cdot\vec{n}\,dS$: $x^2+y^2-z=0 \longrightarrow \vec{\nabla}f = (2x)\vec{i} + (2y)\vec{j} + (-1)\vec{k}$,

$|\vec{\nabla}f| = \sqrt{(2x)^2+(2y)^2+(-1)^2} = \sqrt{4x^2+4y^2+1}$,

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2x)\,\vec{i} + (2y)\,\vec{j} + (-1)\,\vec{k}}{\sqrt{4x^2+4y^2+1}} ;$$

$$\sec\gamma = \frac{|\vec{\nabla} F|}{|f_z|} = \frac{\sqrt{4x^2+4y^2+1}}{1} ; \text{ then}$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iint_{S_2} \frac{\cancel{2xy} - \cancel{2xy} + xz}{\sqrt{4x^2+4y^2+1}}\, dS$$

$$= \iint_R \frac{xz}{\sqrt{4x^2+4y^2+1}} \cdot \sec\gamma\, dA$$

$$= \iint_R \frac{xz}{\sqrt{4x^2+4y^2+1}} \cdot \frac{\sqrt{4x^2+4y^2+1}}{1}\, dA$$

$$= \iint_R x\,(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^1 (r\cos\theta)\, r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^4 \cos\theta\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{5} r^5 \cos\theta \Big|_{r=0}^{r=1} \right) d\theta$$

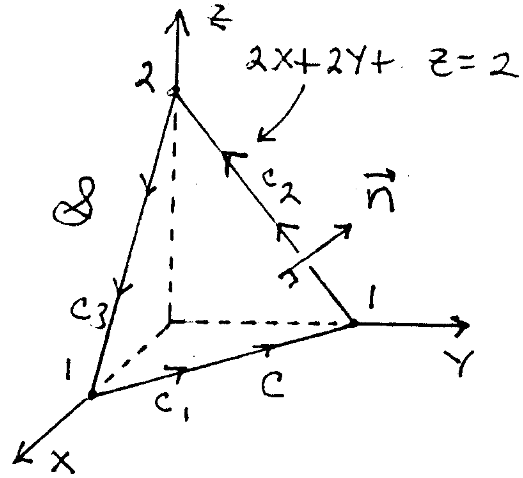
$$= \int_0^{2\pi} \frac{1}{5}\cos\theta\, d\theta = \frac{1}{5}\sin\theta \Big|_0^{2\pi} = \frac{1}{5}\overset{0}{\cancel{\sin 2\pi}} - \frac{1}{5}\overset{0}{\cancel{\sin 0}} = 0$$

so

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{S_1} \vec{F}\cdot\vec{n}\, dS + \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = 0 + 0 = 0 ;$$

this verifies Divergence Theorem.

8.) (14 pts.) Verify Stoke's Theorem for $\vec{F}(x,y,z) = (x)\,\vec{i} + (y)\,\vec{j} + (xz)\,\vec{k}$, where the surface S is that portion of the plane $2x+2y+z=2$ lying in the first octant, and the unit normal vector $\vec{n}$ is pointing upward.



$\vec{\nabla} f = (2)\vec{i} + (2)\vec{j} + (1)\vec{k}$,

$|\vec{\nabla} f| = \sqrt{2^2+2^2+1^2} = 3$, so

$\vec{n} = \left(\frac{2}{3}\right)\vec{i} + \left(\frac{2}{3}\right)\vec{j} + \left(\frac{1}{3}\right)\vec{k}$;

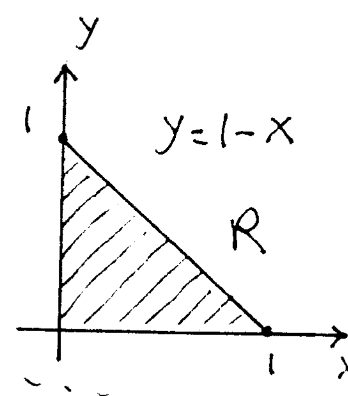
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ X & Y & XZ \end{vmatrix}$$

$$= (0-0)\vec{i} - (z-0)\vec{j} + (0-0)\vec{k} = (-z)\vec{j};$$

Stoke's Theorem: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$:

I.) $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \iint_S \left(-\frac{2}{3}z\right) dS$

$\sec\gamma = \frac{|\vec{\nabla} f|}{|f_z|} = \frac{3}{1} = 3$



$$= \iint_R -\frac{2}{3}(2-2x-2y)\underbrace{\sec\gamma}_{3}\,dA$$

$$= \int_0^1\int_0^{1-x} (-4+4x+4y)\,dy\,dx$$

$$= \int_0^1 (-4y+4xy+2y^2)\Big|_{y=0}^{y=1-x}\,dx$$

$$= \int_0^1 \left(-4(1-x)+4x(1-x)+2(1-x)^2\right)dx$$

$$= \int_0^1 \left(-4+4x+4x-4x^2+2(x^2-2x+1)\right)dx$$

$$= \int_0^1 (-4+8x-4x^2+2x^2-4x+2)\,dx$$

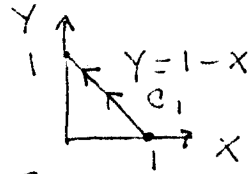
$$= \int_0^1 (-2x^2+4x-2)\,dx = \left(-\frac{2}{3}x^3+2x^2-2x\right)\Big|_0^1$$

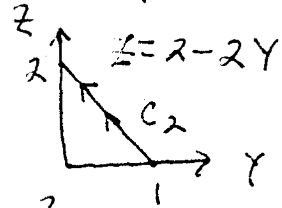
$$= -\frac{2}{3}+2-2 = -\frac{2}{3}$$

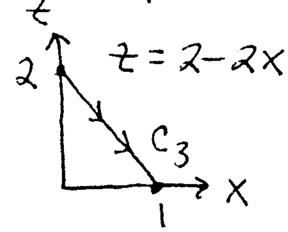
II.) C_1 $(z=0)$: $\begin{cases} x = t \\ y = 1-t \end{cases}$ for $t=1$ to $t=0$

C_2 $(x=0)$: $\begin{cases} y = t \\ z = 2-2t \end{cases}$ for $t=1$ to $t=0$

C_3 $(y=0)$: $\begin{cases} x = t \\ z = 2-2t \end{cases}$ for $t=0$ to $t=1$







$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} M\,dx + N\,dy + P\,dz$$

$$= \int_1^0 \left(x\cdot\frac{dx}{dt} + y\,\frac{dy}{dt} + x\cancelto{0}{z}\,\frac{dz}{dt}\right)dt$$

$$= \int_1^0 \big(t(1) + (1-t)(-1)\big)\,dt = \int_1^0 (t-1+t)\,dt$$

$$= \int_1^0 (2t-1)\,dt = (t^2-t)\Big|_1^0 = 0-0 = 0;$$

$$\int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} M\,dx + N\,dy + P\,dz$$

$$= \int_1^0 \left(\cancelto{0}{x}\cdot\frac{dx}{dt} + y\cdot\frac{dy}{dt} + \cancelto{0}{x}z\,\frac{dz}{dt}\right)dt = \int_1^0 t\cdot(1)\,dt = \frac{1}{2}t^2\Big|_1^0 = \frac{-1}{2};$$

$$\int_{C_3} \vec{F}\cdot\vec{T}\,ds = \int_{C_3} M\,dx + N\,dy + P\,dz$$

$$= \int_0^1 \left(x\cdot\frac{dx}{dt} + \cancelto{0}{y}\,\frac{dy}{dt} + xz\,\frac{dz}{dt}\right)dt = \int_0^1 \big(t(1) + t(2-2t)(-2)\big)\,dt$$

$$= \int_0^1 (t - 4t + 4t^2)\,dt = \int_0^1 (4t^2 - 3t)\,dt$$

$$= \left(\frac{4}{3}t^3 - \frac{3}{2}t^2\right)\Big|_0^1 = \frac{4}{3} - \frac{3}{2} = \frac{-1}{6}\ ; \text{ then}$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds + \int_{C_3} \vec{F}\cdot\vec{T}\,ds$$

$$= (0) + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) = \frac{-3}{6} - \frac{1}{6} = \frac{-4}{6} = \frac{-2}{3};$$

this verifies Stoke's Theorem.

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Assume that curve C is given parametrically by $\vec{r}(t) = (f(t))\,\vec{i} + (g(t))\,\vec{j} + (h(t))\,\vec{k}$ for $t \geq 0$. Let $s = s(t)$ be the arc length of curve C from $t = 0$ to t. Assume that the unit tangent vector is given by

$$\vec{T}(t) = \vec{T}(t(s)) = (s)\,\vec{i} + (s^2)\,\vec{j} + (s^3)\,\vec{k}\,.$$

Find the curvature of C when the arc length is $s = 1$.

$$K = \left| \frac{dT}{ds} \right| = \left| (1)\vec{i} + (2s)\vec{j} + (3s^2)\vec{k} \right|$$

$$= \sqrt{(1)^2 + (2s)^2 + (3s^2)^2}$$

$$= \sqrt{1 + 4s^2 + 9s^4}$$; if $s = 1$, then

curvature $K = \sqrt{14}$.