Math 21D
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Path Independence & Conservative Vector Fields

<u>Defn</u> Let $\vec{F}$ be a vector field defined on region D & let A & B be any two points in D. If

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

for any two paths C_1 & C_2 from point A to point B, then we call $\vec{F}$ a <u>conservative</u> <u>vector</u> <u>field</u>. We also call $\int_C \vec{F}\cdot\vec{T}\,ds$ <u>path</u> <u>independent</u> <u>in</u> <u>D</u> for all curves C that trace a path from A to B.

<u>Defn</u> If $\vec{F}$ is a vector field & there exists a scalar function f such that $\vec{F}$ is gradient field of f (i.e $\vec{F} = \nabla f$) $\Rightarrow$ f is called a <u>potential</u> <u>function</u> <u>of</u> <u>$\vec{F}$</u>.

Fundamental Theorem for Line Integrals

Let $f(x,y,z)$ be a scalar function & let $\vec{\nabla} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$ be its gradient field defined on path $C: \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ for $a \le t \le b$.
Let $A = \vec{r}(a)$ & $B = \vec{r}(b)$

$$\Rightarrow \int_C \vec{\nabla} f\cdot\vec{T}\,ds = f(\vec{r}(t))\Big|_a^b = f(B) - f(A)$$

or

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f\cdot d\vec{r} = f(B) - f(A)$$

Thm 1 Let $\vec{F}$ be vector field defined on region D. $\vec{F}$ is conservative iff $\vec{F}$ is a gradient field of some function f (i.e. $\vec{F} = \nabla f$)

Thm 2 A vector field $\vec{F}$ is conservative iff $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed carve C.

Note: Combining Thm 1 & Thm 2, we have the following equivalencies:

$\vec{F} = \vec{\nabla} f$ on D (There exists potential function of $\vec{F}$) $\iff$ $\vec{F}$ conservative on D $\iff$ $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ $\forall$ closed curves C in D

Component Test for Conservative Fields

Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field $\Rightarrow$ $\vec{F}$ is conservative iff

$$P_y = N_z, \quad M_z = P_x, \quad \& \quad N_x = M_y$$