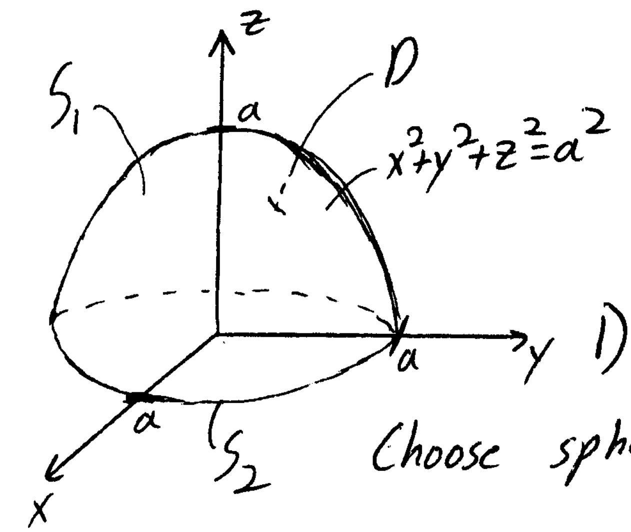
Math 21D
Vogler

Divergence Theorem Example

Ex Verify the Divergence Theorem for $\vec{F}(x,y,z) = (z^2+2)\vec{k}$ for D: top half of sphere $x^2+y^2+z^2=a^2$ with base the disk $x^2+y^2 \le a^2$ in xy-plane.



Divergence Thm

$$\underset{(2)}{\iint_S \vec{F}\cdot\vec{n}\, d\sigma} = \underset{(1)}{\iiint_D \operatorname{div} \vec{F}\, dV}$$

$$\vec{F} = (z^2+2)\vec{k} \Rightarrow \operatorname{div}\vec{F} = 0+0+2z = 2z$$

1) $\iiint_D \operatorname{div}\vec{F}\, dV = \iiint_D 2z\, dV$

Choose spherical coords: $D: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \\ 0 \le \rho \le a \end{cases}$

$$\Rightarrow \iiint_D 2z\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 2\rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 2\rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = 2\int_0^{2\pi}\int_0^{\pi/2} \frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=a} \cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \frac{1}{2}a^4 \int_0^{2\pi}\int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta = \frac{1}{2}a^4 \int_0^{2\pi} \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{\phi=\pi/2} d\theta$$

$$= \frac{1}{4}a^4\int_0^{2\pi} d\theta = \frac{1}{2}a^4\pi$$

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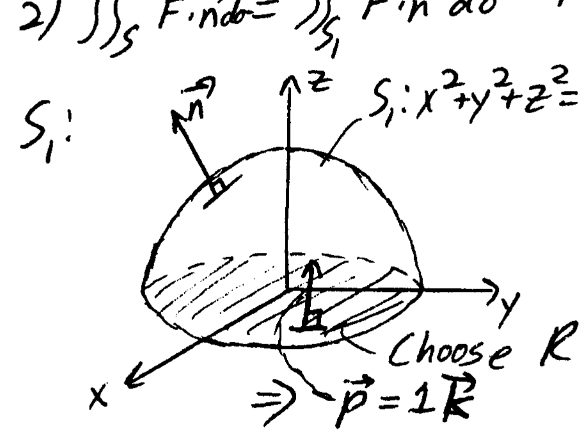
2) $\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma$ where S_1: $\frac{1}{2}$ sphere, S_2: Disk

S_1:



Let $f(x,y,z) = x^2+y^2+z^2$ (Implicit Surface)

$$\Rightarrow \vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\vec{\nabla} f| = \sqrt{4x^2+4y^2+4z^2} = \sqrt{4a^2} = 2a$$

$$\Rightarrow \vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}$$

'Exchange Rate': $\dfrac{|\vec{\nabla} f|}{|\vec{\nabla} f\cdot \vec{p}|} = \dfrac{|\vec{\nabla} f|}{|\vec{\nabla} f\cdot \vec{k}|} = \dfrac{2a}{|2z|} = \dfrac{a}{z}$

Note: $\vec{\nabla} f\cdot \vec{k} = (2x\vec{i} + 2y\vec{j} + 2z\vec{k})\cdot(0\vec{i} + 0\vec{j} + 1\vec{k}) = 0+0+2z = 2z$

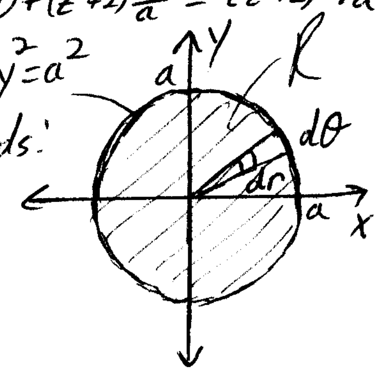
$\vec{F}\cdot\vec{n} = (0\vec{i} + 0\vec{j} + (z^2+2)\vec{k})\cdot(\frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}) = 0+0+(z^2+2)\frac{z}{a} = (z^2+2)\,z/a$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} (z^2+2)\cdot\frac{z}{a}\,d\sigma$$

$$= \iint_R (z^2+2)\,\frac{\cancel{z}}{\cancel{a}}\cdot\frac{\cancel{a}}{\cancel{z}}\,dA = \iint_R (z^2+2)\,dA$$

Choose Polar coords:

$R: \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le a$



$$= \int_0^{2\pi}\int_0^a (a^2 - r^2 + 2)\,r\,dr\,d\theta$$

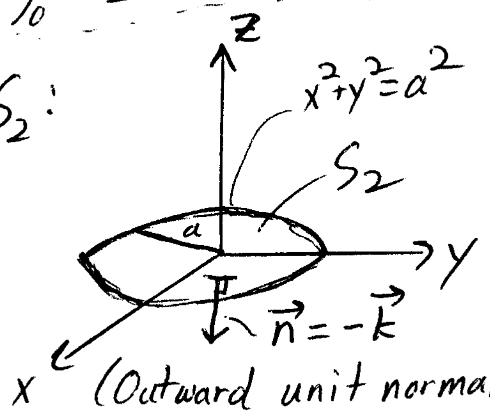
$$= \int_0^{2\pi}\int_0^a (a^2+2)r - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(a^2+2)r^2 - \frac{1}{4}r^4\Big|_{r=0}^{r=a}\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(a^2+2)a^2 - \frac{1}{4}a^4\,d\theta = \left(\frac{1}{4}a^4 + a^2\right)\int_0^{2\pi} d\theta = \frac{1}{2}\pi a^4 + 2\pi a^2$$

Note: $x^2+y^2+z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 - y^2$
$\Rightarrow z^2 = a^2 - r^2$

S_2:



Note: $\vec{F}\cdot\vec{n} = (0\vec{i} + 0\vec{j} + (z^2+2)\vec{k})\cdot(0\vec{i} + 0\vec{j} - 1\vec{k})$
$= -(z^2+2)$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_2} -(z^2+2)\,d\sigma$$

(Since S_2 is already 'flat', this is just a double integral)

$$\iint_{S_2} -(z^2+2)\,d\sigma = \iint_{S_2} -(0+2)\,d\sigma = -2\iint_{S_2} d\sigma = -2(\text{Area of } S_2)$$

$= -2\pi a^2$

Hence, $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma$

$= \frac{1}{2}\pi a^4 + \cancel{2\pi a^2} - \cancel{2\pi a^2} = \frac{1}{2}\pi a^4 = \iiint_D \operatorname{div}\vec{F}\,dV$ ✓

Stoke's