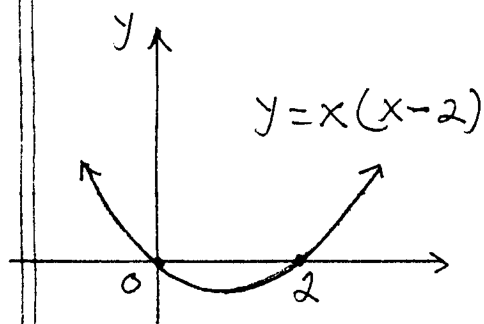
Section 13.1

1.) $\vec{r}(t) = (t+1)\vec{i} + (t^2-1)\vec{j} \rightarrow$

$\begin{cases} x = t+1 \rightarrow t = x-1 \\ y = t^2-1 \end{cases}$ (SUB) $\rightarrow y = (x-1)^2 - 1 \rightarrow$

$y = x^2 - 2x + \cancel{1} - \cancel{1} \rightarrow y = x(x-2)$ (parabola);



$\vec{v}(t) = \vec{r}\,'(t) = 1 \cdot \vec{i} + 2t \cdot \vec{j} \xrightarrow{D}$

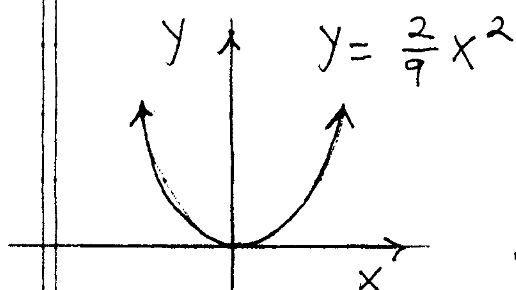
$\vec{a}(t) = \vec{v}\,'(t) = 0 \cdot \vec{i} + 2 \cdot \vec{j}$; then

$\vec{v}(1) = \vec{i} + 2\vec{j}$ and

$\vec{a}(1) = 2\vec{j}$

3.) $\vec{r}(t) = e^t\vec{i} + \frac{2}{9}e^{2t}\vec{j} \rightarrow$

$\begin{cases} x = e^t \\ y = \frac{2}{9}e^{2t} = \frac{2}{9}(e^t)^2 \end{cases}$ (SUB) $\rightarrow y = \frac{2}{9}x^2$ (parabola);



$\vec{v}(t) = \vec{r}\,'(t) = e^t \cdot \vec{i} + \frac{4}{9}e^{2t}\vec{j} \xrightarrow{D}$

$\vec{a}(t) = \vec{v}\,'(t) = e^t \cdot \vec{i} + \frac{8}{9}e^{2t} \cdot \vec{j}$; then

$v(\ln 3) = e^{\ln 3} \cdot \vec{i} + \frac{4}{9} \cdot e^{2\ln 3} \cdot \vec{j}$

$= 3\vec{i} + \frac{4}{9}e^{\ln 3^2} \cdot \vec{j} = 3\vec{i} + \frac{4}{9} \cdot 9\vec{j} \rightarrow \vec{v} = 3\vec{i} + 4\vec{j}$,

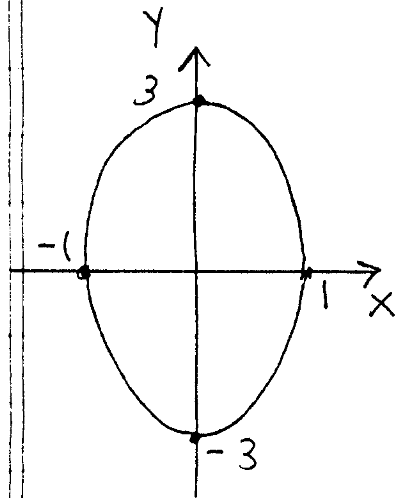
$\vec{a}(\ln 3) = e^{\ln 3}\vec{i} + \frac{8}{9}e^{2\ln 3} \cdot \vec{j} = 3\vec{i} + 8\vec{j}$

4.) $\vec{r}(t) = (\cos 2t)\vec{i} + (3\sin 2t) \cdot \vec{j} \rightarrow$

$\begin{cases} x = \cos 2t \\ y = 3\sin 2t \rightarrow \frac{y}{3} = \sin 2t \end{cases}$, then

$x^2 + \left(\frac{y}{3}\right)^2 = \cos^2 2t + \sin^2 2t = 1 \rightarrow x^2 + \left(\frac{y}{3}\right)^2 = 1$;

(ellipse) $\vec{v}(t) = (-2\sin 2t)\vec{i} + (6\cos 2t)\vec{j}$



$\xrightarrow{D}$ $\vec{a}(t) = (-4\cos 2t)\vec{i} + (-12\sin 2t)\vec{j}$,

then

$\vec{v}(0) = (-2\overset{0}{\cancel{\sin 0}})\vec{i} + (6\overset{1}{\cancel{\cos 0}})\vec{j}$

$= 6\vec{j}$

$\vec{a}(0) = (-4\overset{1}{\cancel{\cos 0}})\vec{i} + (-12\overset{0}{\cancel{\sin 0}})\vec{j}$

$= -4\vec{i}$

6.) $\vec{r}(t) = (4\cos\frac{t}{2})\vec{i} + (4\sin\frac{t}{2})\vec{j}$ $\xrightarrow{D}$

$\vec{v}(t) = (-2\sin\frac{t}{2})\vec{i} + (2\cos\frac{t}{2})\vec{j}$ $\xrightarrow{D}$

$\vec{a}(t) = (-\cos\frac{t}{2})\vec{i} + (-\sin\frac{t}{2})\vec{j}$; if $t = \pi$

then $\vec{r}(\pi) = 0\cdot\vec{i} + 4\vec{j} = 4\vec{j}$,

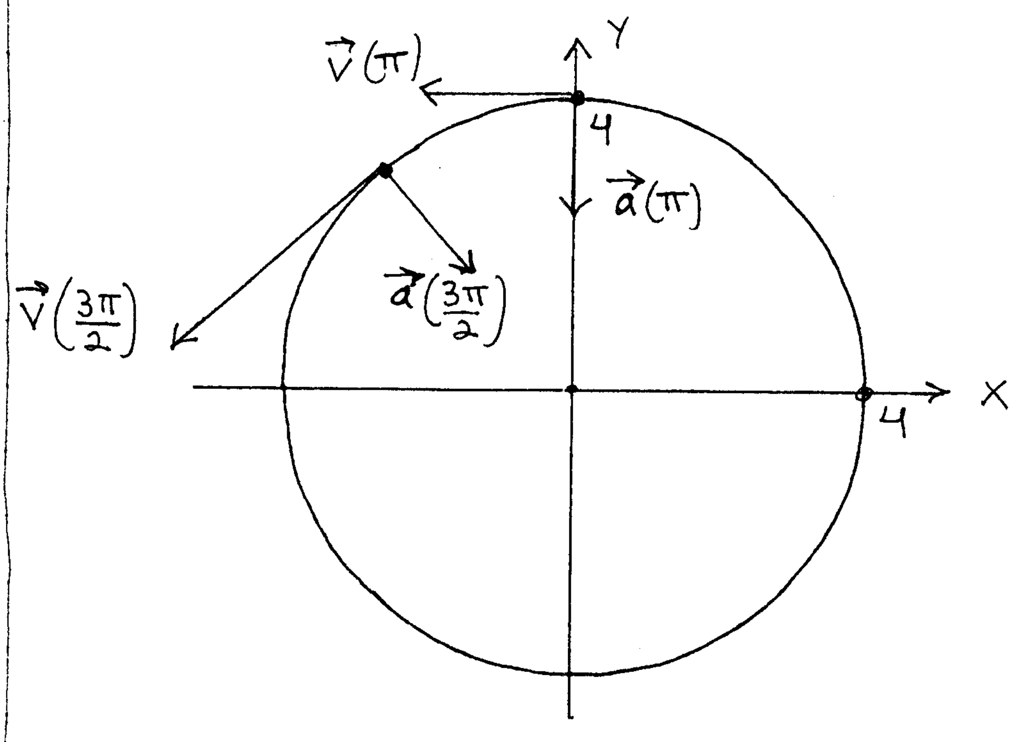
$\vec{v}(\pi) = -2\vec{i} + 0\cdot\vec{j} = -2\vec{i}$,

$\vec{a}(\pi) = 0\cdot\vec{i} + (-1)\vec{j} = -\vec{j}$; if $t = \frac{3\pi}{2}$

then $\vec{r}\left(\frac{3\pi}{2}\right) = \left(4\cdot\frac{-\sqrt{2}}{2}\right)\vec{i} + \left(4\cdot\frac{\sqrt{2}}{2}\right)\vec{j} = -2\sqrt{2}\,\vec{i} + 2\sqrt{2}\,\vec{j}$,

$\vec{v}\left(\frac{3\pi}{2}\right) = \left(-2\cdot\frac{\sqrt{2}}{2}\right)\vec{i} + \left(2\cdot\frac{-\sqrt{2}}{2}\right)\vec{j} = -\sqrt{2}\cdot\vec{i} - \sqrt{2}\cdot\vec{j}$,

$\vec{a}\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}\vec{i} + \frac{-\sqrt{2}}{2}\vec{j}$;



8.) $\vec{r}(t) = t\cdot\vec{i} + (t^2+1)\cdot\vec{j} \xrightarrow{D}$

$\vec{v}(t) = 1\cdot\vec{i} + 2t\cdot\vec{j} \xrightarrow{D}$

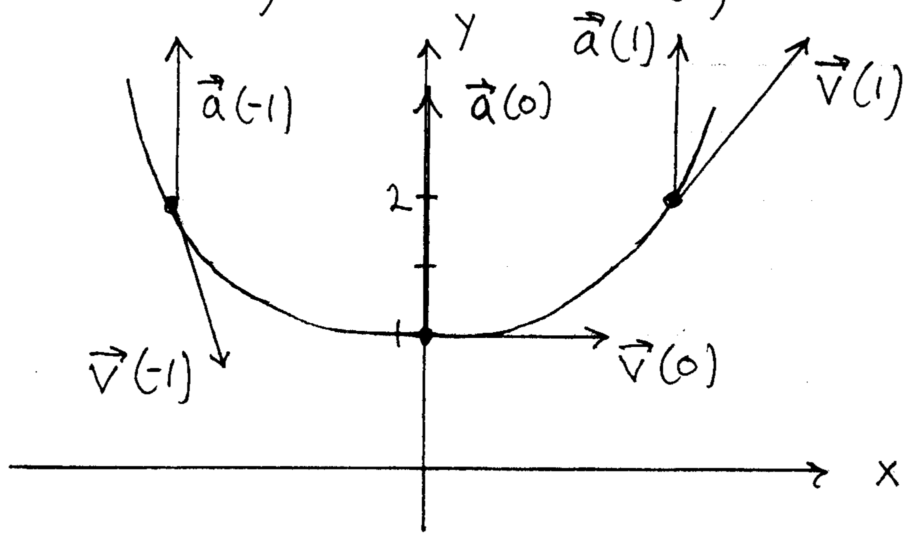
$\vec{a}(t) = 0\cdot\vec{i} + 2\cdot\vec{j}$; if $t=-1 \rightarrow$

$\vec{r}(-1) = -\vec{i} + 2\vec{j}$,

$\vec{v}(-1) = \vec{i} - 2\vec{j}$, $\vec{a}(-1) = 2\vec{j}$; if $t=0 \rightarrow$

$\vec{r}(0) = \vec{j}$, $\vec{v}(0) = \vec{i}$, $\vec{a}(0) = 2\cdot\vec{j}$; if $t=1 \rightarrow$

$\vec{r}(1) = \vec{i} + 2\vec{j}$, $\vec{v}(1) = \vec{i} + 2\vec{j}$, $\vec{a}(1) = 2\vec{j}$



10.) $\vec{r}(t) = (1+t)\vec{i} + \frac{1}{\sqrt{2}}t^2\cdot\vec{j} + \frac{1}{3}t^3\cdot\vec{k} \xrightarrow{D}$

$\vec{v}(t) = 1\cdot\vec{i} + \sqrt{2}\,t\cdot\vec{j} + t^2\vec{k} \xrightarrow{D}$

$\vec{a}(t) = 0\cdot\vec{i} + \sqrt{2}\cdot\vec{j} + 2t\cdot\vec{k}$; if $t=1$ then

$\vec{v}(1) = \vec{i} + \sqrt{2}\,\vec{j} + \vec{k}$ so speed is

$|\vec{v}(1)| = \sqrt{1^2 + (\sqrt{2})^2 + 1^2} = \sqrt{4} = 2$, then

direction of motion is

$\frac{\vec{v}(1)}{|\vec{v}(1)|} = \underline{\frac{1}{2}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{2}\vec{k}}$ and

$\vec{v}(1) = 2\left(\frac{1}{2}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{2}\vec{k}\right)$

12.) $\vec{r}(t) = (\sec t)\vec{i} + (\tan t)\vec{j} + \frac{4}{3}t\vec{k} \xrightarrow{D}$

$\vec{v}(t) = (\sec t \tan t)\vec{i} + (\sec^2 t)\vec{j} + \frac{4}{3}\vec{k} \xrightarrow{D}$

$\vec{a}(t) = (\sec t \cdot \sec^2 t + \sec t \tan t \cdot \tan t)\cdot\vec{i}$
$+ (2\sec t \cdot \sec t \tan t)\vec{j} + 0 \cdot \vec{k}$

$= (\sec^3 t + \sec t \cdot \tan^2 t)\vec{i}$
$+ (2\sec^2 t \tan t)\vec{j}$; if $t = \pi/6$

$\vec{v}(\pi/6) = (\sec\frac{\pi}{6}\tan\frac{\pi}{6})\vec{i} + (\sec^2\frac{\pi}{6})\vec{j} + \frac{4}{3}\vec{k}$

$= \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\vec{i} + \left(\frac{2}{\sqrt{3}}\right)^2\vec{j} + \frac{4}{3}\vec{k}$

$= \frac{2}{3}\vec{i} + \frac{4}{3}\vec{j} + \frac{4}{3}\vec{k}$, so speed is

$|\vec{v}(\frac{\pi}{6})| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{16}{9}}$

$= \sqrt{\frac{36}{9}} = \sqrt{4} = 2$; and direction

of motion is $\frac{\vec{v}(\pi/6)}{|\vec{v}(\pi/6)|} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$;

$\vec{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$.

13.) $\vec{r}(t) = (2\ln(t+1))\vec{i} + t^2\vec{j} + \frac{t^2}{2}\vec{k} \xrightarrow{D}$

$\vec{v}(t) = \frac{2}{t+1}\vec{i} + 2t\vec{j} + t\vec{k} \xrightarrow{D}$

$\vec{a}(t) = \frac{-2}{(t+1)^2}\vec{i} + 2\vec{j} + 1\vec{k}$; if $t=1$

$\vec{v}(1) = 1\vec{i} + 2\vec{j} + 1\vec{k}$ so speed is

$|\vec{v}(1)| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$ and direction

of motion is

$$\frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k} \quad ;$$

$$\vec{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right) \quad .$$

15.) $\vec{r}(t) = (3t+1)\vec{i} + \sqrt{3}\cdot t\vec{j} + t^2\vec{k} \xrightarrow{D}$

$\vec{v}(t) = 3\cdot\vec{i} + \sqrt{3}\,\vec{j} + 2t\,\vec{k} \xrightarrow{D}$

$\vec{a}(t) = 0\cdot\vec{i} + 0\cdot\vec{j} + 2\vec{k}$; if $t=0$ then

$\vec{v}(0) = 3\vec{i} + \sqrt{3}\,\vec{j} + 0\vec{k}$ and

$\vec{a}(0) = 0\cdot\vec{i} + 0\cdot\vec{j} + 2\vec{k} \rightarrow$

$|\vec{v}(0)| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12}$ and

$|\vec{a}(0)| = \sqrt{2^2} = 2$; then

$$\cos\theta = \frac{\vec{v}(0)\bullet\vec{a}(0)}{|\vec{v}(0)|\,|\vec{a}(0)|} = \frac{0+0+0}{\sqrt{12}\cdot 2} = 0 \quad \text{so}$$

$\theta = \frac{\pi}{2}$.

17.) $\vec{r}(t) = (\ln(t^2+1))\vec{i} + (\arctan t)\vec{j} + \sqrt{t^2+1}\cdot\vec{k} \xrightarrow{D}$

$$\vec{v}(t) = \frac{2t}{t^2+1}\vec{i} + \frac{1}{1+t^2}\vec{j} + \frac{t}{\sqrt{t^2+1}}\vec{k} \xrightarrow{D}$$

$$\vec{a}(t) = \frac{(t^2+1)(2) - 2t(2t)}{(t^2+1)^2}\vec{i} + \frac{-2t}{(1+t^2)^2}\vec{j}$$

$$+ \frac{\sqrt{t^2+1}\,(1) - t\cdot\frac{1}{\cancel{2}}(t^2+1)^{-1/2}(\cancel{2}t)}{t^2+1}\vec{k}$$

$$= \frac{2-2t^2}{(t^2+1)^2}\vec{i} + \frac{-2t}{(1+t^2)^2}\vec{j} + \frac{1}{(t^2+1)^{3/2}}\vec{k} \; ; \text{ if } t=0$$

$\vec{v}(0) = 0\vec{i} + 1\cdot\vec{j} + 0\cdot\vec{k}$ and
$\vec{a}(0) = 2\vec{i} + 0\cdot\vec{j} + 1\cdot\vec{k}$ so that
$|\vec{v}(0)| = \sqrt{1^2} = 1$ and $|\vec{a}(0)| = \sqrt{2^2+1^2} = \sqrt{5}$,
then

$$\cos\theta = \frac{\vec{v}(0)\cdot\vec{a}(0)}{|\vec{v}(0)|\,|\vec{a}(0)|} = \frac{0+0+0}{(1)(\sqrt{5})} = 0$$

so $\theta = \frac{\pi}{2}$.

19.) $\vec{r}(t) = (\sin t)\vec{i} + (t^2 - \cos t)\vec{j} + e^t\vec{k} \xrightarrow{D}$
$\vec{r}\,'(t) = (\cos t)\vec{i} + (2t + \sin t)\vec{j} + e^t\vec{k}$
and $t = 0 \rightarrow$ point of tangency
is $(\sin 0, 0^2 - \cos 0, e^0) = (0, -1, 1)$ and
tangent vector is
$\vec{r}\,'(0) = (\cos 0)\vec{i} + (2(0) + \sin 0)\vec{j} + e^0\vec{k}$
$= 1\cdot\vec{i} + 0\cdot\vec{j} + 1\cdot\vec{k}$ so
tangent line is given by

$$L: \begin{cases} x = 0 + (1)t \\ y = -1 + (0)t \\ z = 1 + (1)t \end{cases} \rightarrow \begin{cases} x = t \\ y = -1 \\ z = 1 + t \end{cases} \quad \text{for } -\infty < t < \infty$$

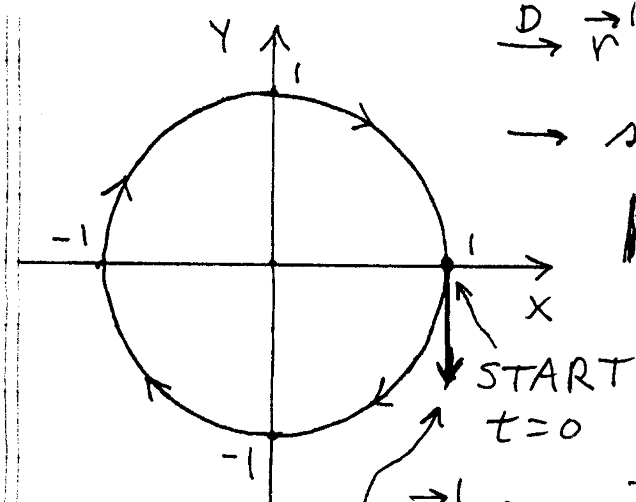
22.) $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (\sin 2t)\vec{k} \xrightarrow{D}$
$\vec{r}\,'(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + (2\cos 2t)\vec{k}$
and $t = \frac{\pi}{2} \rightarrow$ point of tangency is
$(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}, \sin\pi) = (0,1,0)$ and
tangent vector is

$$\vec{r}\,'(\tfrac{\pi}{2}) = (-\sin\tfrac{\pi}{2})\vec{i} + (\cos\tfrac{\pi}{2})\vec{j} + (2\cos\pi)\vec{k}$$
$$= -1\cdot\vec{i} + 0\cdot\vec{j} + {}^-2\cdot\vec{k}\text{ , so}$$

tangent line is given by

$$L: \begin{cases} x = 0 + (-1)t \\ y = 1 + (0)t \\ z = 0 + (-2)t \end{cases} \rightarrow \begin{cases} x = -t \\ y = 1 \\ z = {}^-2t \end{cases} \text{ for } -\infty < t < \infty$$

23.) d.) $\vec{r}(t) = (\cos t)\vec{i} + (-\sin t)\vec{j}$ for $t \geq 0$



$\xrightarrow{D} \vec{r}\,'(t) = (-\sin t)\vec{i} + (-\cos t)\vec{j}$
$\rightarrow$ speed is

$$|\vec{r}\,'(t)|^2 = \sqrt{(-\sin t)^2 + (-\cos t)^2}$$
$$= \sqrt{\sin^2 t + \cos^2 t}$$
$$= \sqrt{1} = 1 \text{ , and}$$

$\vec{r}\,'(0) = 0\cdot\vec{i} + (-1)\vec{j}$

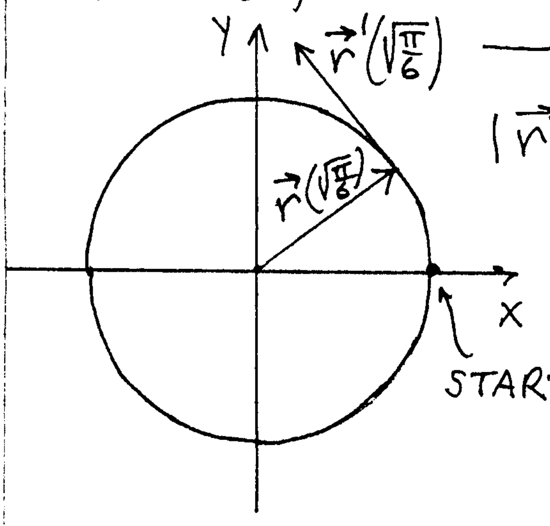
i.) YES, speed = 1

ii.) YES, since $\vec{r}\,'(t)$ has constant length we know $\frac{d}{dt}(\vec{r}\,'(t)) = \vec{r}\,''(t)$ is $\perp$ to $\vec{r}\,'(t)$ (Example from class)

iii.) The particle moves clockwise since $\vec{r}\,'(0) = -\vec{j}$.

23.) e.) $\vec{r}(t) = \cos(t^2)\cdot\vec{i} + \sin(t^2)\cdot\vec{j}$ for $t \geq 0$

$\rightarrow \vec{r}\,'(t) = -2t\sin(t^2)\cdot\vec{i} + 2t\cos(t^2)\cdot\vec{j}$



$\longrightarrow$ speed is

$$|\vec{r}\,'(t)| = \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2}$$
$$= \sqrt{4t^2\sin^2(t^2) + 4t^2\cos^2(t^2)}$$
$$= \sqrt{4t^2(\sin^2(t^2) + \cos^2(t^2))}$$
$$= \sqrt{4t^2(1)} = 2|t| = 2t;$$

$\vec{r}\,'(0) = 0\cdot\vec{i} + 0\cdot\vec{j}$ (no good information)

$$\vec{r}\,'(\sqrt{\tfrac{\pi}{6}}) = (-2\sqrt{\tfrac{\pi}{6}}\,\overset{1/2}{\cancel{\sin\tfrac{\pi}{6}}})\vec{i} + (2\sqrt{\tfrac{\pi}{6}}\,\overset{\sqrt{3}/2}{\cancel{\cos\tfrac{\pi}{6}}})\vec{j}$$
$$= -\sqrt{\tfrac{\pi}{6}}\,\vec{i} + \sqrt{3}\sqrt{\tfrac{\pi}{6}}\,\vec{j}$$

i.) The speed of motion is $2t$, not a constant speed.

ii.) No, since velocity vector $\vec{r}'(t)$ does not have constant length

iii.) The particle moves counter-clockwise (SEE $\vec{r}'(\sqrt{\pi/6})$).

24.) $\vec{r}(t) = (2\vec{i} + 2\vec{j} + \vec{k})$
$+ \cos t \left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\right)$
$+ \sin t \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right)$

$= \left(2 + \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t\right)\vec{i}$
$+ \left(2 - \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t\right)\vec{j}$
$+ \left(1 + \frac{1}{\sqrt{3}}\sin t\right)\vec{k} \longrightarrow$

$$\begin{cases} x = 2 + \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t \\ y = 2 - \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t \\ z = 1 + \frac{1}{\sqrt{3}}\sin t \end{cases} \longrightarrow z - 1 = \frac{1}{\sqrt{3}}\sin t$$

$$\rightarrow \begin{cases} x = 2 + \frac{1}{\sqrt{2}}\cos t + (z-1) \\ y = 2 - \frac{1}{\sqrt{2}}\cos t + (z-1) \end{cases} \quad \text{(ADD)}$$

$\rightarrow x + y = 4 + 2z - 2$

$\rightarrow \boxed{x + y - 2z = 2}$ (a plane);

now find distance between (x, y, z) and $(2, 2, 1)$:

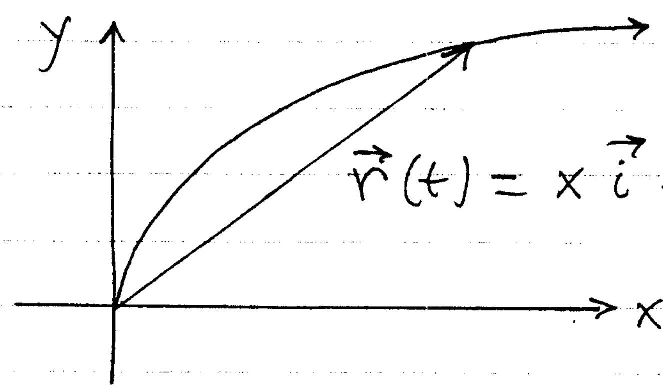
$$L = \sqrt{(x-2)^2+(y-2)^2+(z-1)^2}$$

$$= \sqrt{\left(\tfrac{1}{\sqrt2}\cos t+\tfrac{1}{\sqrt3}\sin t\right)^2 + \left(-\tfrac{1}{\sqrt2}\cos t+\tfrac{1}{\sqrt3}\sin t\right)^2 + \left(\tfrac{1}{\sqrt3}\sin t\right)^2}$$

$$= \sqrt{\tfrac12\cos^2 t+\cancel{\tfrac{2}{\sqrt6}\cos t\sin t}+\tfrac13\sin^2 t + \tfrac12\cos^2 t-\cancel{\tfrac{2}{\sqrt6}\cos t\sin t}+\tfrac13\sin^2 t + \tfrac13\sin^2 t}$$

$$= \sqrt{\cos^2 t+\sin^2 t} = \sqrt{1} = 1 \,;$$

thus points (x,y,z) lie in the plane $x+y-2z=2$ and are 1 unit away from point $(2,2,1)$, a circle of radius 1 centered at $(2,2,1)$.

25.) 

$y^2 = 2x$ or $x = \frac12 y^2$

$\vec{r}(t) = x\,\vec{i} + y\,\vec{j} = x(t)\,\vec{i} + y(t)\,\vec{j}$

then

velocity vector is

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$\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j}$ so

speed $= |\vec{r}\,'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = 5$

$\rightarrow \boxed{(x'(t))^2 + (y'(t))^2 = 25}$; and

$\boxed{(y(t))^2 = 2x(t)} \xrightarrow{D} 2 \cdot y(t)\, y'(t) = 2x'(t)$

$\rightarrow \boxed{(y(t))^2 (y'(t))^2 = (x'(t))^2}$; then

(SUB) $2x(t)\left(25 - (x'(t))^2\right) = (x'(t))^2$

$\rightarrow$ (Solve for $x'(t)$.) $\longrightarrow$

$50\,x(t) - 2x(t) \cdot (x'(t))^2 = (x'(t))^2 \rightarrow$

$50\,x(t) = 2x(t) \cdot (x'(t))^2 + (x'(t))^2 \rightarrow$

$50\,x(t) = (2x(t) + 1)(x'(t))^2 \longrightarrow$

$x'(t)^2 = \dfrac{50x(t)}{2x(t)+1} \longrightarrow \boxed{x'(t) = \sqrt{\dfrac{50x(t)}{2x(t)+1}}}$;

if $x = 2$, then

$x'(t) = \sqrt{20} = \boxed{2\sqrt{5}}$ and

$y'(t) = \sqrt{25 - (x'(t))^2} = \sqrt{25 - 20} = \boxed{\sqrt{5}}$;

thus at $(2,2)$ velocity is

$\vec{r}\,(t) = x'(t) \cdot \vec{i} + y'(t)\,\vec{j}$

$= 2\sqrt{5}\,\vec{i} + \sqrt{5}\,\vec{j}$.

34.) Theorem: If $\vec{u}(t) = a\vec{i} + b\vec{j} + c\vec{k}$, a constant vector, then $\vec{u}'(t) = 0\cdot\vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k}$.

Proof: $\vec{u}(t) = a\vec{i} + b\vec{j} + c\vec{k} \xrightarrow{D}$

$$\frac{d}{dt}\vec{u}(t) = \frac{d}{dt}(a)\vec{i} + \frac{d}{dt}(b)\vec{j} + \frac{d}{dt}(c)\vec{k}$$

$$= 0\cdot\vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k}.$$

QED.