

Section 16.2

1.) $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \rightarrow$

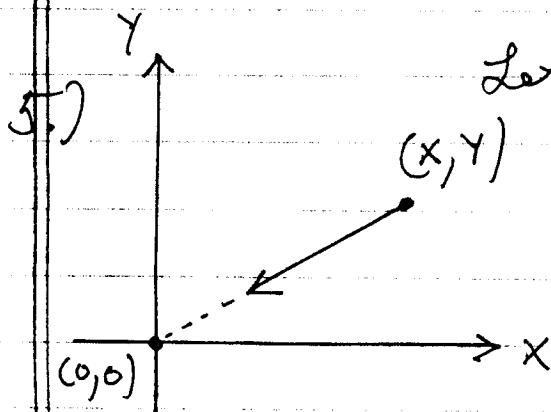
$$\begin{aligned}\vec{\nabla} f(x, y, z) &= \frac{-1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2x \vec{i} \\ &\quad + \frac{-1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2y \vec{j} + \frac{-1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2z \vec{k} \\ &= \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \vec{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \vec{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \vec{k}\end{aligned}$$

3.) $g(x, y, z) = e^z - \ln(x^2 + y^2) \rightarrow$

$$\begin{aligned}\vec{\nabla} g(x, y, z) &= \frac{-1}{x^2 + y^2} \cdot 2x \vec{i} + \frac{-1}{x^2 + y^2} \cdot 2y \vec{j} + e^z \vec{k} \\ &= \frac{-2x}{x^2 + y^2} \vec{i} + \frac{-2y}{x^2 + y^2} \vec{j} + e^z \vec{k}\end{aligned}$$

4.) $g(x, y, z) = xy + yz + xz \rightarrow$

$$\vec{\nabla} g(x, y, z) = (y + z) \vec{i} + (x + z) \vec{j} + (x + y) \vec{k}$$



Let $\vec{v} = (0 - x) \vec{i} + (0 - y) \vec{j}$
 $= (-x) \vec{i} + (-y) \vec{j}$ and

$$\begin{aligned}|\vec{v}| &= \sqrt{(-x)^2 + (-y)^2} \\ &= \sqrt{x^2 + y^2}\end{aligned}$$

so unit vector is $\vec{u} = \frac{-x}{\sqrt{x^2 + y^2}} \vec{i} + \frac{-y}{\sqrt{x^2 + y^2}} \vec{j}$;

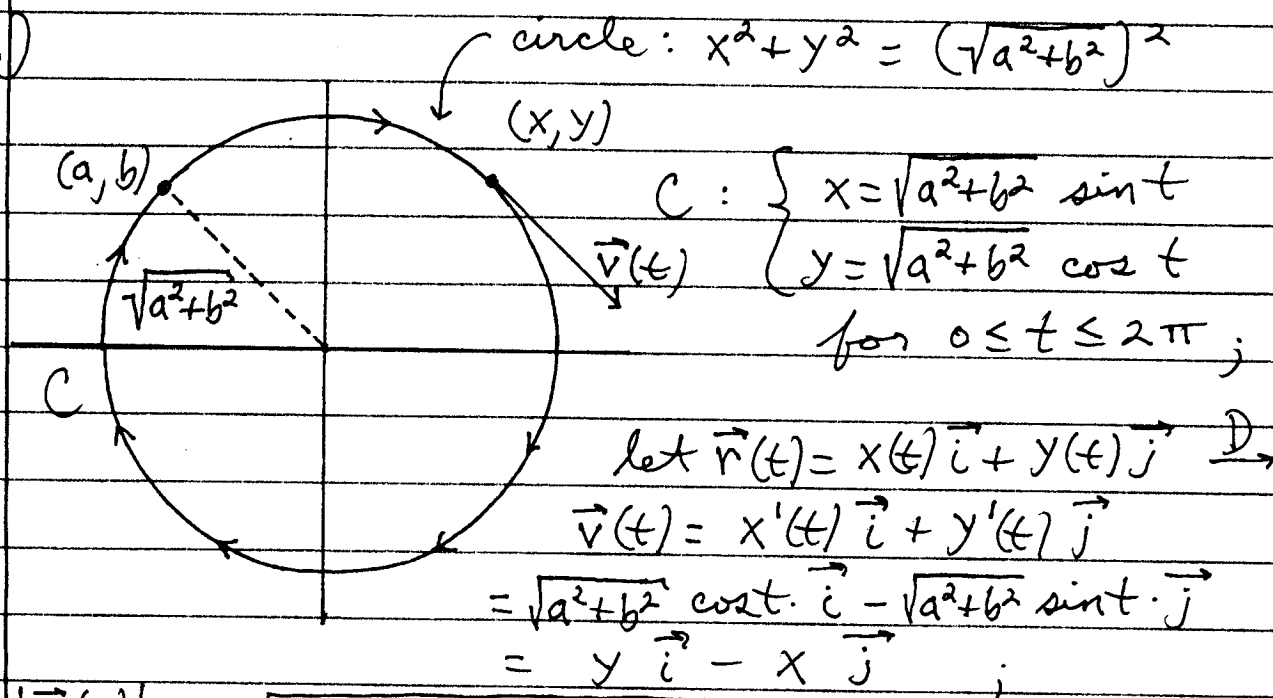
magnitude of $\vec{F}$ is

$|\vec{F}(x,y)| = k \cdot \frac{1}{x^2+y^2}$ so vector field is

$$\vec{F}(x,y) = k \cdot \frac{1}{x^2+y^2} \cdot \vec{u} \quad (k \text{ is a constant.})$$

$$= \frac{-kx}{(x^2+y^2)^{3/2}} \vec{i} + \frac{-ky}{(x^2+y^2)^{3/2}} \vec{j}$$

6.)



$$|\vec{v}(t)| = \sqrt{(\sqrt{a^2+b^2} \cos t)^2 + (-\sqrt{a^2+b^2} \sin t)^2}$$

$$= \sqrt{(a^2+b^2)(\cos^2 t + \sin^2 t)} = \sqrt{a^2+b^2};$$

let

$$\vec{F}(x,y) = \vec{v}(t) = y \vec{i} - x \vec{j}$$

7.) $\vec{F}(x,y,z) = (3y)\vec{i} + (2x)\vec{j} + (4z)\vec{k}$

a.) $C_1: \vec{r}(t) = (t)\vec{i} + (t)\vec{j} + (t)\vec{k}$ for $0 \leq t \leq 1$

$$\underline{D} \rightarrow \vec{r}'(t) = (1)\vec{i} + (1)\vec{j} + (1)\vec{k} \text{ and}$$

$$\vec{F}(x, y, z) = \vec{F}(t, t, t) = (3t)\vec{i} + (2t)\vec{j} + (4t)\vec{k},$$

$$\text{so } \int_C \vec{F} \cdot \vec{T} ds = \int_{C_1} \vec{F} \cdot \vec{T} ds = \int_{C_1} \vec{F} \cdot \vec{r}'(t) dx$$

$$= \int_0^1 [(3t)(1) + (2t)(1) + (4t)(1)] dx$$

$$= \int_0^1 9t dx = 9 \cdot \frac{1}{2} t^2 \Big|_0^1 = \frac{9}{2} (1)^2 = \frac{9}{2}$$

$$b.) C_2: \vec{r}(t) = (t)\vec{i} + (t^2)\vec{j} + (t^4)\vec{k} \text{ for } 0 \leq t \leq 1$$

$$\underline{D} \rightarrow \vec{r}'(t) = (1)\vec{i} + (2t)\vec{j} + (4t^3)\vec{k} \text{ and}$$

$$\vec{F}(x, y, z) = \vec{F}(t, t^2, t^4)$$

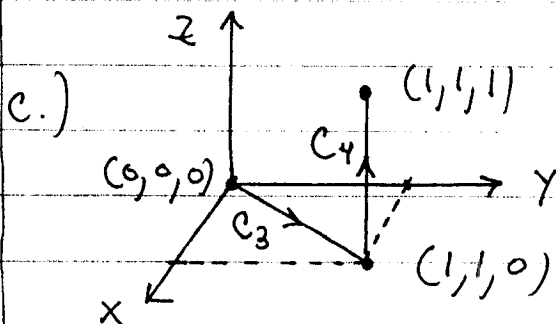
$$= (3t^2)\vec{i} + (2t)\vec{j} + (4t^4)\vec{k}$$

$$\text{so } \int_C \vec{F} \cdot \vec{T} ds = \int_{C_2} \vec{F} \cdot \vec{T} ds = \int_{C_2} \vec{F} \cdot \vec{r}'(t) dx$$

$$= \int_0^1 [(3t^2)(1) + (2t)(2t) + (4t^4)(4t^3)] dx$$

$$= \int_0^1 (3t^2 + 4t^2 + 16t^7) dx = \int_0^1 (7t^2 + 16t^7) dx$$

$$= \left(\frac{7}{3} t^3 + 2t^8 \right) \Big|_0^1 = \frac{7}{3} + \frac{6}{3} = \frac{13}{3}$$



$$C_3: \begin{cases} x=t \\ y=t \\ z=0 \end{cases} \text{ for } 0 \leq t \leq 1,$$

$$C_4: \begin{cases} x=1 \\ y=1 \\ z=t \end{cases} \text{ for } 0 \leq t \leq 1.$$

$$\vec{r}_3(t) = (t)\vec{i} + (t)\vec{j} \rightarrow \vec{r}_3'(t) = (1)\vec{i} + (1)\vec{j} \quad \text{for } 0 \leq t \leq 1 \text{ and}$$

$$\vec{F}(x, y, z) = \vec{F}(t, t, 0) = (3t)\vec{i} + (2t)\vec{j} + (0)\vec{k}$$

$$\vec{r}_4(t) = (1)\vec{i} + (1)\vec{j} + (t)\vec{k} \xrightarrow{D} \vec{r}_4'(t) = (0)\vec{i} + (0)\vec{j} + (1)\vec{k} \quad \text{for } 0 \leq t \leq 1 \text{ and}$$

$$\vec{F}(x, y, z) = \vec{F}(1, 1, t) = (3)\vec{i} + (2)\vec{j} + (4t)\vec{k}; \text{ then}$$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_{C_3} \vec{F} \cdot \vec{T} ds + \int_{C_4} \vec{F} \cdot \vec{T} ds$$

$$= \int_{C_3} \vec{F} \cdot \vec{r}_3'(t) dt + \int_{C_4} \vec{F} \cdot \vec{r}_4'(t) dt$$

$$= \int_0^1 [(3t)(1) + (2t)(1)] dt + \int_0^1 [(4t)(1)] dt$$

$$= \int_0^1 5t dt + \int_0^1 4t dt = \frac{5}{2} t^2 \Big|_0^1 + 2t^2 \Big|_0^1$$

$$= \frac{5}{2} + \frac{4}{2} = \frac{9}{2}$$

12.) $\vec{F}(x, y, z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

a.) $C_1: \vec{r}(t) = (t)\vec{i} + (t)\vec{j} + (t)\vec{k} \text{ for } 0 \leq t \leq 1$

$$\xrightarrow{D} \vec{r}'(t) = (1)\vec{i} + (1)\vec{j} + (1)\vec{k} \text{ and}$$

$$\begin{aligned} \vec{F}(x, y, z) &= \vec{F}(t, t, t) = (t+t)\vec{i} + (t+t)\vec{j} + (t+t)\vec{k} \\ &= (2t)\vec{i} + (2t)\vec{j} + (2t)\vec{k} \end{aligned}$$

so

$$\int_C \vec{F} \cdot \vec{T} ds = \int_{C_1} \vec{F} \cdot \vec{T} ds = \int_{C_1} \vec{F} \cdot \vec{r}'(t) dt$$

$$= \int_0^1 [(2t)(1) + (2t)(1) + (2t)(1)] dt = \int_0^1 6t dt$$

$$= 3t^2 \Big|_0^1 = 3$$

b.) $C_2: \vec{r}(t) = (t)\vec{i} + (t^2)\vec{j} + (t^4)\vec{k}$ for $0 \leq t \leq 1$
 $\xrightarrow{D} \vec{r}'(t) = (1)\vec{i} + (2t)\vec{j} + (4t^3)\vec{k}$ and
 $\vec{F}(x,y,z) = \vec{F}(t, t^2, t^4)$
 $= (t^2 + t^4)\vec{i} + (t + t^4)\vec{j} + (t + t^2)\vec{k};$

then

$$\int_{C_2} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot \vec{T} ds = \int_{C_2} \vec{F} \cdot \vec{r}'(t) dt$$

$$= \int_0^1 [(t^2 + t^4)(1) + (t + t^4)(2t) + (t + t^2)(4t^3)] dt$$

$$= \int_0^1 [t^2 + t^4 + 2t^2 + 2t^5 + 4t^4 + 4t^5] dt$$

$$= \int_0^1 [3t^2 + 5t^4 + 6t^5] dt$$

$$= (t^3 + t^5 + t^5) \Big|_0^1 = 1 + 1 + 1 = 3$$

c.) SEE solution 7.) c.)

$\vec{r}_3(t) = (t)\vec{i} + (t)\vec{j} \xrightarrow{D} \vec{r}_3'(t) = (1)\vec{i} + (1)\vec{j}$ for $0 \leq t \leq 1$
and $\vec{F}(x,y,z) = \vec{F}(t,t,0) = (t+0)\vec{i} + (t+0)\vec{j} + (t+t)\vec{k}$
 $= (t)\vec{i} + (t)\vec{j} + (2t)\vec{k}$
for $0 \leq t \leq 1 \rightarrow \vec{r}_4(t) = (1)\vec{i} + (1)\vec{j} + (t)\vec{k} \xrightarrow{D} \vec{r}_4'(t) = (1)\vec{k}$ and
 $\vec{F}(x,y,z) = \vec{F}(1,1,t) = (1+t)\vec{i} + (1+t)\vec{j} + (1+1)\vec{k}$
 $= (1+t)\vec{i} + (1+t)\vec{j} + (2)\vec{k};$ then

$$\begin{aligned}
\int_C \vec{F} \cdot \vec{T} ds &= \int_{C_3} \vec{F} \cdot \vec{T} ds + \int_{C_4} \vec{F} \cdot \vec{T} ds \\
&= \int_{C_3} \vec{F} \cdot \vec{r}'_3(t) dt + \int_{C_4} \vec{F} \cdot \vec{r}'_4(t) dt \\
&= \int_0^1 [(t)(1) + (t)(1) + (2t)(0)] dt \\
&\quad + \int_0^1 [(1+t)(0) + (1+t)(0) + (2)(1)] dt \\
&= \int_0^1 2t dt + \int_0^1 2 dt = t^2 \Big|_0^1 + 2t \Big|_0^1 \\
&= 1 + 2 = 3.
\end{aligned}$$

19.) $\vec{F}(x, y, z) = (xy)\vec{i} + (y)\vec{j} + (-yz)\vec{k}$
 (1) $\vec{r}(t) = (t)\vec{i} + (t^2)\vec{j} + (t)\vec{k}$ for $0 \leq t \leq 1$ $\xrightarrow{D}$
 $\vec{r}'(t) = (1)\vec{i} + (2t)\vec{j} + (1)\vec{k}$, and
 $\vec{F}(x, y, z) = \vec{F}(t, t^2, t)$
 $= (t \cdot t^2)\vec{i} + (t^2)\vec{j} + (-t^2 \cdot t)\vec{k}$
 $= (t^3)\vec{i} + (t^2)\vec{j} + (-t^3)\vec{k}$; then

$$\begin{aligned}
\text{work} &= \int_C \vec{F} \cdot \vec{T} ds = \int_C \vec{F} \cdot \vec{r}'(t) dt \\
&= \int_0^1 [(t^3)(1) + (t^2)(2t) + (-t^3)(1)] dt \\
&= \int_0^1 [t^3 + 2t^3 - t^3] dt = \int_0^1 2t^3 dt \\
&= \frac{1}{2} t^4 \Big|_0^1 = \frac{1}{2}
\end{aligned}$$

$$21.) \vec{F}(x, y, z) = (z)\vec{i} + (x)\vec{j} + (y)\vec{k}$$

$$C: \vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + (t)\vec{k}$$

$$\text{for } 0 \leq t \leq 2\pi \xrightarrow{D}$$

$$\vec{r}'(t) = (\cos t)\vec{i} + (-\sin t)\vec{j} + (1)\vec{k}, \text{ and}$$

$$\vec{F}(x, y, z) = \vec{F}(\sin t, \cos t, t)$$

$$= (t)\vec{i} + (\sin t)\vec{j} + (\cos t)\vec{k}; \text{ then}$$

$$\text{Work} = \int_C \vec{F} \cdot \vec{T} ds = \int_C \vec{F} \cdot \vec{r}'(t) dt$$

$$= \int_0^{2\pi} [t \cos t - \sin^2 t + \cos t] dt$$

$$= \int_0^{2\pi} t \cos t dt - \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) dt + \sin t \Big|_0^{2\pi}$$

$$(\text{let } u = t, dv = \cos t dt \\ \rightarrow du = dt, v = \sin t)$$

$$= t \sin t \Big|_0^{2\pi} - \int_0^{2\pi} \sin t dt$$

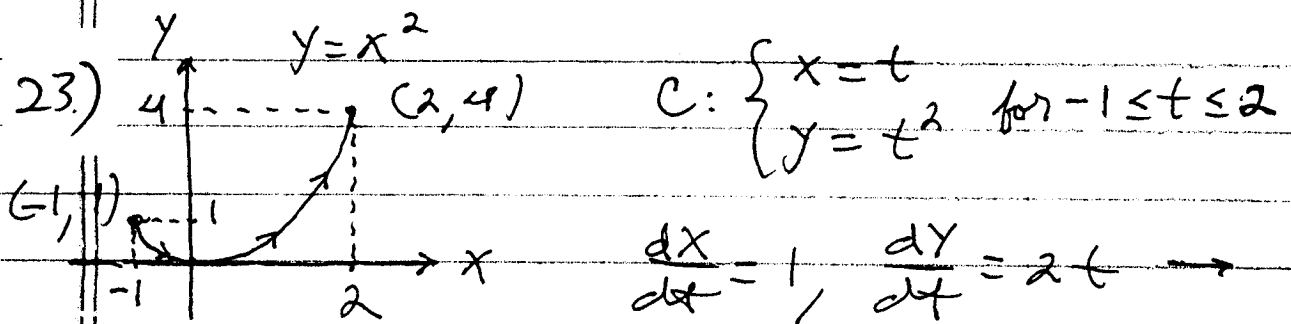
$$+ \frac{-1}{2} \left(t - \frac{1}{2} \sin 2t \right) \Big|_0^{2\pi} + \sin t \Big|_0^{2\pi}$$

$$= 2\pi \sin 2\pi - (0) \sin 0 + \cos t \Big|_0^{2\pi}$$

$$- \frac{1}{2} \left[\left(2\pi - \frac{1}{2} \sin 4\pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$+ \cos 2\pi - \cos 0$$

$$= \cos 2\pi - \cos 0 - \pi = -\pi$$



$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t \rightarrow$$

$$\int_C xy \, dx + (x+y) \, dy$$

$$= \int_{-1}^2 \left[(t)(t^2) \frac{dx}{dt} + (t+t^2) \frac{dy}{dt} \right] dt$$

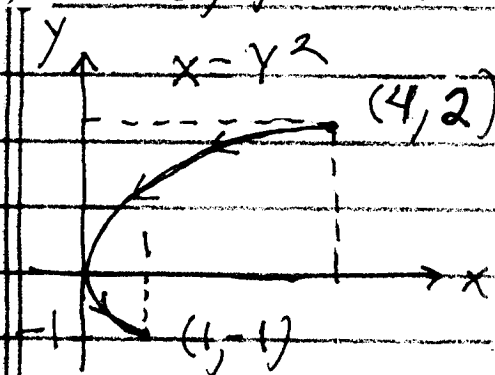
$$= \int_{-1}^2 \left[t^3 \cdot (1) + (t+t^2)(2t) \right] dt$$

$$= \int_{-1}^2 (3t^3 + 2t^2) dt = \left(\frac{3}{4}t^4 + \frac{2}{3}t^3 \right) \Big|_{-1}^2$$

$$= \left(12 + \frac{16}{3} \right) - \left(\frac{3}{4} - \frac{2}{3} \right)$$

$$= \frac{144}{12} + \frac{64}{12} - \frac{9}{12} + \frac{8}{12} = \frac{207}{12} = \frac{69}{4}$$

25) $\vec{F}(x,y) = (x^2)\vec{i} + (-y)\vec{j}$ and



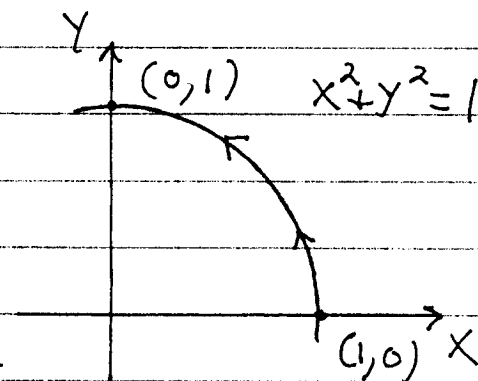
$$C: \begin{cases} x = t^2 \\ y = t \end{cases} \text{ for } t = -1 \text{ to } t = 2$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C M \, dx + N \, dy$$

$$= \int_C \left[x^2 \cdot \frac{dx}{dt} + (-y) \frac{dy}{dt} \right] dt$$

$$\begin{aligned}
 &= \int_2^{-1} [(t^2)^2 \cdot (2t) + (-t)(1)] dt \\
 &= \int_2^{-1} [2t^5 - t] dt = \left(\frac{1}{3} t^6 - \frac{1}{2} t^2 \right) \Big|_2^{-1} \\
 &= \left(\frac{1}{3} - \frac{1}{2} \right) - \left(\frac{64}{3} - 2 \right) = -\frac{63}{3} + \frac{3}{2} \\
 &= -21 + \frac{3}{2} = -\frac{42}{2} + \frac{3}{2} = -\frac{39}{2}
 \end{aligned}$$

26.) $\vec{F}(x,y) = (y)\vec{i} + (-x)\vec{j}$



$$C: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \text{ for } 0 \leq t \leq \frac{\pi}{2}$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t;$$

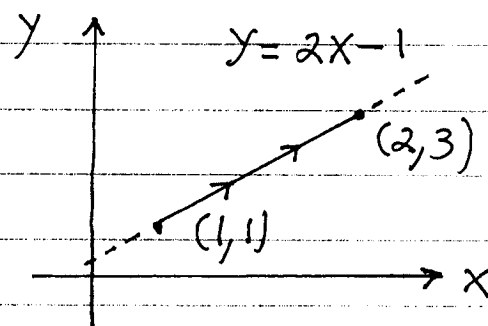
$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$$

$$= \int_C \left[y \cdot \frac{dx}{dt} + (-x) \frac{dy}{dt} \right] dt$$

$$= \int_0^{\frac{\pi}{2}} [\sin t \cdot (-\sin t) + (-\cos t) \cdot \cos t] dt$$

$$= \int_0^{\frac{\pi}{2}} -(\underbrace{\sin^2 t + \cos^2 t}_1) dt = -t \Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{2}$$

27.) $\vec{F}(x,y) = (xy)\vec{i} + (y-x)\vec{j}$



$$C: \begin{cases} x=t \\ y=2t-1 \end{cases} \text{ for } 1 \leq t \leq 2 ;$$

$$\text{Work} = \int_C \vec{F} \cdot \vec{T} \, ds = \int_C M \, dx + N \, dy$$

$$= \int_C \left[xy \cdot \frac{dx}{dt} + (y-x) \frac{dy}{dt} \right] dt$$

$$= \int_C [(t)(2t-1)(1) + ((2t-1)-t)(2)] dt$$

$$= \int_1^2 [2t^2 - t + 2t - 2] dt = \int_1^2 [2t^2 + t - 2] dt$$

$$= \left(\frac{2}{3} t^3 + \frac{1}{2} t^2 - 2t \right) \Big|_1^2$$

$$= \left(\frac{16}{3} + 2 - 4 \right) - \left(\frac{2}{3} + \frac{1}{2} - 2 \right)$$

$$= \frac{14}{3} - \frac{1}{2} = \frac{25}{6}$$

28.) $f(x,y) = (x+y)^2 \rightarrow \vec{\nabla} f = f_x \vec{i} + f_y \vec{j} \rightarrow$

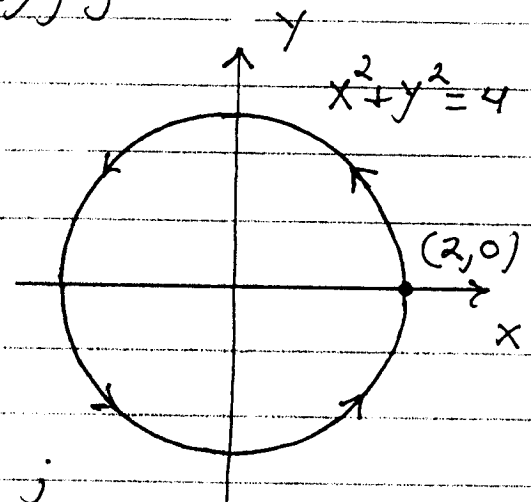
$$\vec{\nabla} f(x,y) = 2(x+y) \vec{i} + 2(x+y) \vec{j}$$

so let

$$\vec{F}(x,y) = 2(x+y) \vec{i} + 2(x+y) \vec{j} ;$$

$$C: \begin{cases} x=2\cos t \\ y=2\sin t \end{cases} \text{ for } 0 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = -2\sin t, \quad \frac{dy}{dt} = 2\cos t ;$$



$$\text{Work} = \int_C \vec{F} \cdot \vec{T} ds = \int_C M dx + N dy$$

$$= \int_C \left[2(x+y) \frac{dx}{dt} + 2(x+y) \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} \left[2(2\cos t + 2\sin t) \cdot -2\sin t + 2(2\cos t + 2\sin t) \cdot 2\cos t \right] dt$$

$$= \int_0^{2\pi} (-8\sin t \cos t - 8\sin^2 t + 8\cos^2 t + 8\sin t \cos t) dt$$

$$= \int_0^{2\pi} 8 \cdot (\cos^2 t - \sin^2 t) dt$$

$$= \int_0^{2\pi} 8 \cdot \cos 2t dt = 8 \cdot \frac{1}{2} \sin 2t \Big|_0^{2\pi}$$

$$= 4 \sin 4\pi - 4 \sin 0 = 0$$

39.) $\underline{(x,y)} \quad \underline{\vec{F}(x,y)}$

$$(2,0) \quad \vec{j}$$

$$(\sqrt{2}, \sqrt{2}) \quad -\frac{\sqrt{2}}{2} \vec{i} + \frac{\sqrt{2}}{2} \vec{j}$$

$$(0,2) \quad -\vec{i}$$

$$(-\sqrt{2}, \sqrt{2}) \quad -\frac{\sqrt{2}}{2} \vec{i} - \frac{\sqrt{2}}{2} \vec{j}$$

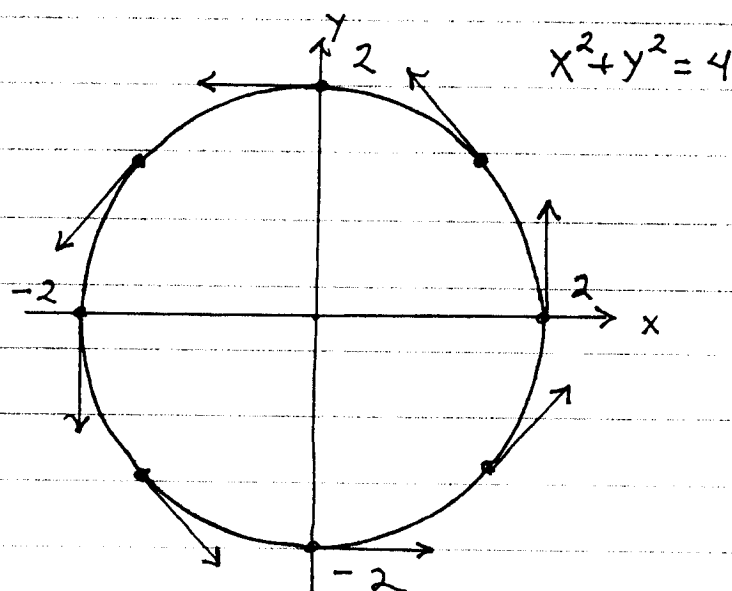
$$(-2,0) \quad -\vec{j}$$

$$(-\sqrt{2}, -\sqrt{2}) \quad \frac{\sqrt{2}}{2} \vec{i} - \frac{\sqrt{2}}{2} \vec{j}$$

$$(0,-2) \quad \vec{i}$$

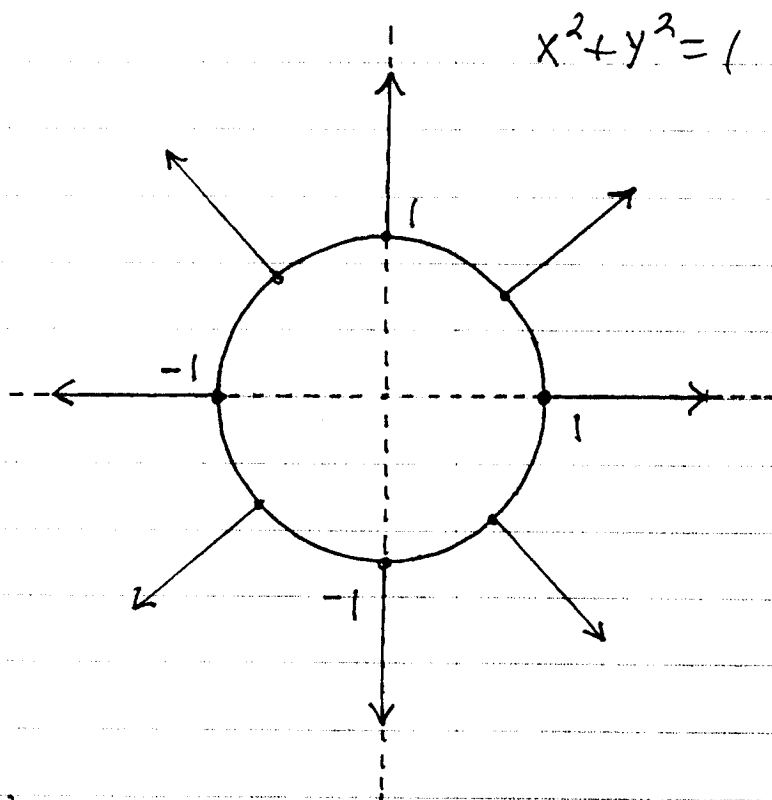
$$(\sqrt{2}, -\sqrt{2}) \quad \frac{\sqrt{2}}{2} \vec{i} + \frac{\sqrt{2}}{2} \vec{j}$$

$$\vec{F}(x,y) = \frac{-y}{\sqrt{x^2+y^2}} \vec{i} + \frac{x}{\sqrt{x^2+y^2}} \vec{j}$$

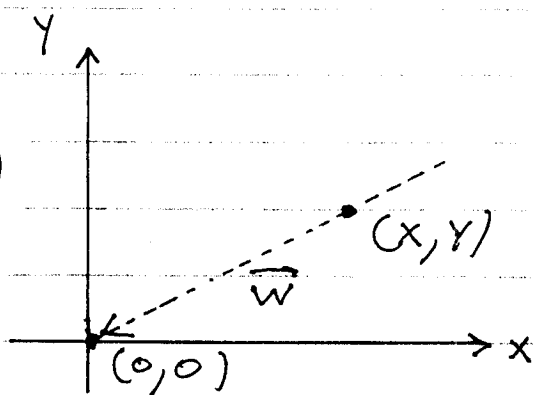


(x, y)	$\vec{F}(x, y)$
$(1, 0)$	$\vec{i}$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$
$(0, 1)$	$\vec{j}$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$
$(-1, 0)$	$-\vec{i}$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$
$(0, -1)$	$-\vec{j}$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$

$$\vec{F}(x, y) = (x)\vec{i} + (y)\vec{j}$$



43.)



The vector at point (x, y)

pointing at origin is

$$\vec{w} = (0 - x)\vec{i} + (0 - y)\vec{j} = (-x)\vec{i} + (-y)\vec{j},$$

its magnitude is

$$|\vec{w}| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2}; \text{ so}$$

unit vector is

$$\vec{F}(x, y) = \frac{\vec{w}}{|\vec{w}|} = \frac{-x}{\sqrt{x^2 + y^2}}\vec{i} + \frac{-y}{\sqrt{x^2 + y^2}}\vec{j}$$