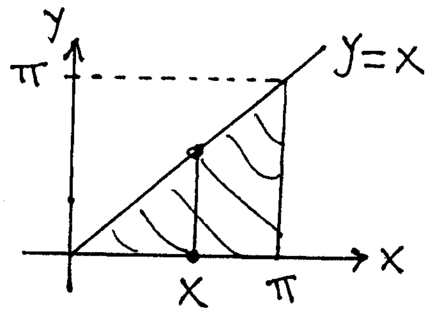
Section 15.2

19.) $\int_0^\pi \int_0^x x \sin y \, dy \, dx$

$= \int_0^\pi \left[x \cdot -\cos y \Big|_{y=0}^{y=x} \right] dx$



$= \int_0^\pi \left(-x \cos x - -x \cdot \overset{1}{\cos 0} \right) dx$

$= \int_0^\pi (x - x \cos x) \, dx = \int_0^\pi x \, dx - \int_0^\pi x \cos x \, dx$

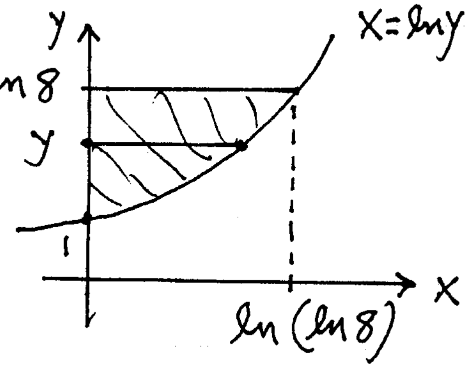
$= \frac{1}{2} x^2 \Big|_0^\pi$

$- \left[x \sin x \Big|_0^\pi - \int_0^\pi \sin x \, dx \right]$ (Let $u = x$, $dv = \cos x \, dx \rightarrow$ $du = dx$, $v = \sin x$)

$= \frac{1}{2}\pi^2 - \left[\left(\pi \overset{0}{\sin \pi} - 0 \overset{0}{\sin 0} \right) - -\cos x \Big|_0^\pi \right]$

$= \frac{1}{2}\pi^2 - \left(\overset{-1}{\cos \pi} - \overset{+1}{\cos 0} \right) = \frac{1}{2}\pi^2 + 2$

21.) $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$



$= \int_1^{\ln 8}\left(e^{x+y}\Big|_{x=0}^{x=\ln y}\right)dy$

$= \int_1^{\ln 8}\left(e^{\ln y + y} - e^y\right)dy$

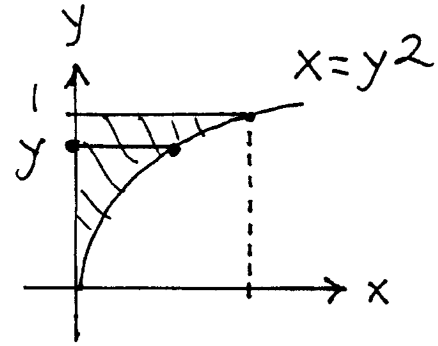
$= \int_1^{\ln 8} e^{\ln y}\cdot e^y\,dy - \int_1^{\ln 8} e^y\,dy$

$= \int_1^{\ln 8} y\cdot e^y\,dy - e^y\Big|_1^{\ln 8}$

(Let $u = y$, $dv = e^y dy$ $\rightarrow du = dy$, $v = e^y$)

$= \left(ye^y\Big|_1^{\ln 8} - \int_1^{\ln 8} e^y dy\right) - \left(e^{\ln 8} - e\right)$

$= \left(\ln 8\cdot e^{\ln 8} - e\right) - e^y\Big|_1^{\ln 8} - 8 + e$

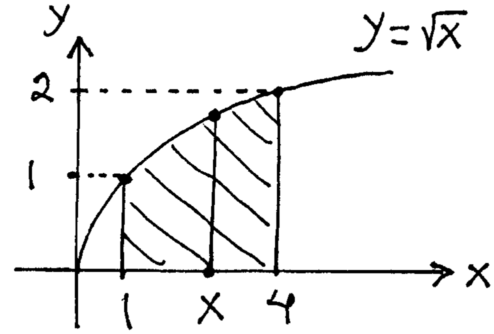


23) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$

$= \int_0^1 \left(3y^2 e^{xy} \Big|_{x=0}^{x=y^2}\right) dy$

$= \int_0^1 (3y^2 e^{y^3} - 3y^2)\,dy = (e^{y^3} - y^3)\Big|_0^1$

$= (e-1) - (1-0) = e-2$



24) $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\,dy\,dx$

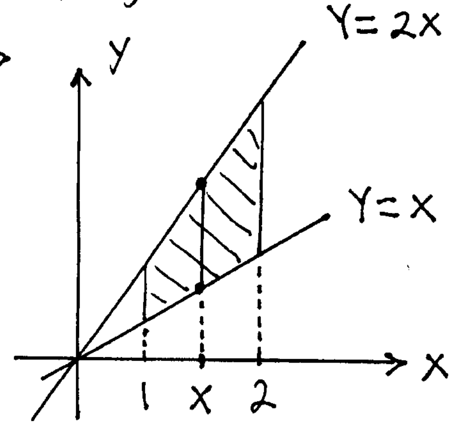
$= \int_1^4 \left(\frac{3}{2}\cdot\sqrt{x}\cdot e^{y/\sqrt{x}} \Big|_{y=0}^{y=\sqrt{x}}\right) dx$

$= \int_1^4 \left(\frac{3}{2}\sqrt{x}\cdot e - \frac{3}{2}\sqrt{x}\right) dx$

$= \left(\cancel{\frac{3}{2}} e \cdot \cancel{\frac{2}{3}} x^{3/2} - \cancel{\frac{3}{2}}\cdot\cancel{\frac{2}{3}} x^{3/2}\right)\Big|_1^4$

$= \left(e(4)^{3/2} - (4)^{3/2}\right) - \left(e(1)^{3/2} - (1)^{3/2}\right)$

$= 8e - 8 - e + 1 = 7e - 7$



25) $\iint_R \frac{x}{y}\,dA = \int_1^2 \int_x^{2x} \frac{x}{y}\,dy\,dx$

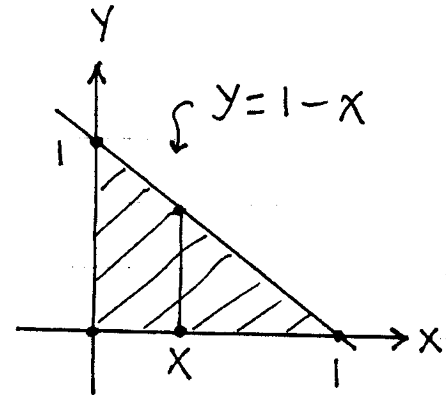
$= \int_1^2 \left(x \ln y \Big|_{y=x}^{y=2x}\right) dx$

$= \int_1^2 (x\ln 2x - x\ln x)\,dx$

$= \int_1^2 \left(x(\ln 2 + \ln x) - x\ln x\right) dx = \int_1^2 \ln 2 \cdot x\,dx$

$= \ln 2 \cdot \frac{1}{2}x^2 \Big|_1^2 = \ln 2\cdot\left(2 - \frac{1}{2}\right) = \frac{3}{2}\ln 2$

26.) $\iint\limits_R (x^2+y^2)\, dA = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$



$= \int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=1-x} dx$

$= \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3\right] dx$

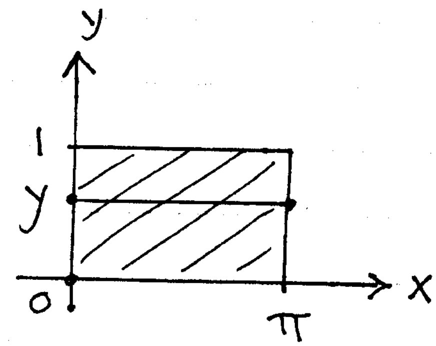
$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x)^3\right] dx$

$= \left(\frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{-1}{12}(1-x)^4\right)\Big|_0^1$

$= \left(\frac{1}{3} - \frac{1}{4} + 0\right) - \left(0 - 0 - \frac{1}{12}\right)$

$= \frac{4}{12} - \frac{3}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$

Extra) $\iint\limits_R y \cos xy \, dA$

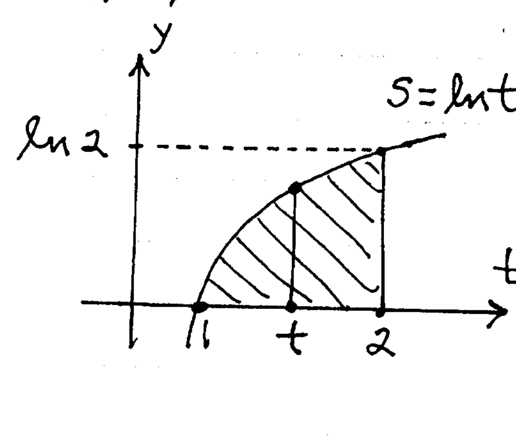


$= \int_0^1 \int_0^{\pi} y \cos xy \, dx\, dy$

$= \int_0^1 \left(\sin xy \Big|_{x=0}^{x=\pi}\right) dy = \int_0^1 (\sin \pi y - \cancel{\sin 0}^{\,0})\, dy$

$= \frac{-1}{\pi} \cos \pi y \Big|_0^1 = \frac{-1}{\pi} \cancel{\cos \pi}^{\,-1} - \frac{-1}{\pi} \cancel{\cos 0}^{\,1}$

$= \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$



28.) $\iint\limits_R e^s \ln t \, dA$

$= \int_1^2 \int_0^{\ln t} e^s \ln t \, ds\, dt$

$$= \int_1^2 \left(e^s \ln t \Big|_{s=0}^{s=\ln t} \right) dt$$

$$= \int_1^2 \left[e^{\ln t} \cdot \ln t - e^0 \ln t \right] dt$$

$$= \int_1^2 (t \ln t - \ln t)\, dt = \int_1^2 (t-1) \ln t \, dt$$

(Let $u = \ln t$, $dv = (t-1)\,dt \rightarrow$

$du = \frac{1}{t}\,dt$, $v = \frac{1}{2}t^2 - t$)

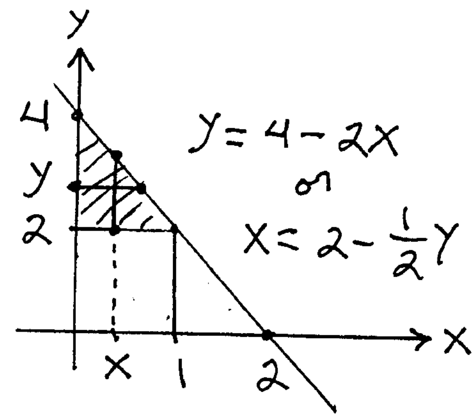
$$= \left(\tfrac{1}{2}t^2 - t\right) \ln t \Big|_1^2 - \int_1^2 \left(\tfrac{1}{2}t - 1\right) dt$$

$$= (0 - 0) - \left(\tfrac{1}{4}t^2 - t\right)\Big|_1^2$$

$$= -(-1) - -\left(\tfrac{1}{4} - 1\right) = 1 + -\tfrac{3}{4} = \tfrac{1}{4}$$

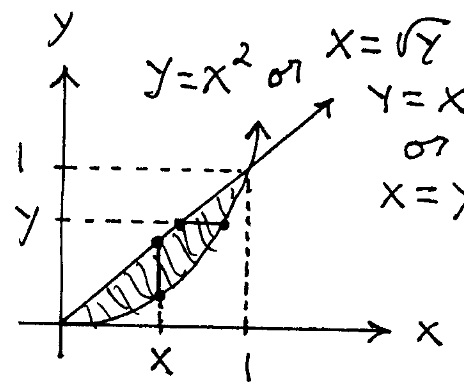
33.) $\int_0^1 \int_2^{4-2x} dy\, dx$

$$= \int_2^4 \int_0^{2-\frac{1}{2}y} dx\, dy$$



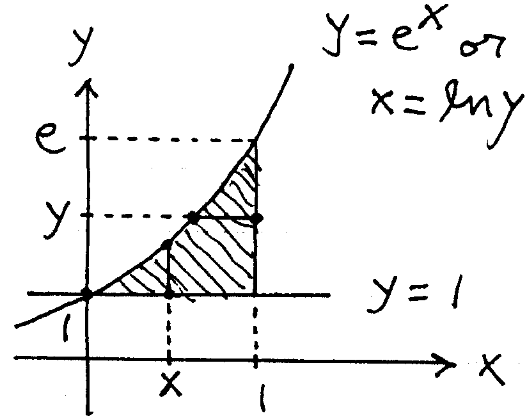
35.) $\int_0^1 \int_y^{\sqrt{y}} dx\, dy$

$$= \int_0^1 \int_{x^2}^{x} dy\, dx$$



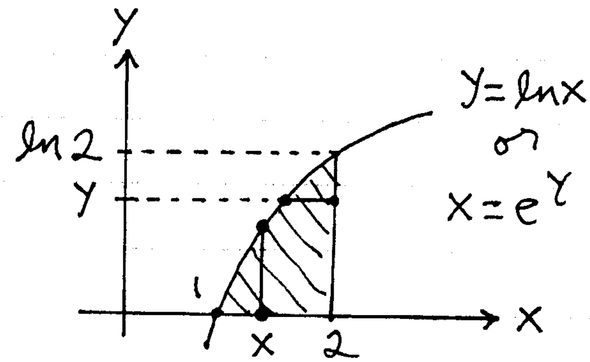
37) $\int_0^1 \int_1^{e^x} dy\, dx$

$= \int_1^e \int_{\ln y}^1 dx\, dy$



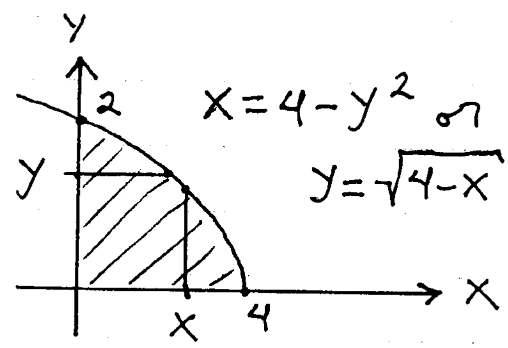
38) $\int_0^{\ln 2} \int_{e^y}^2 dx\, dy$

$= \int_1^2 \int_0^{\ln x} dy\, dx$



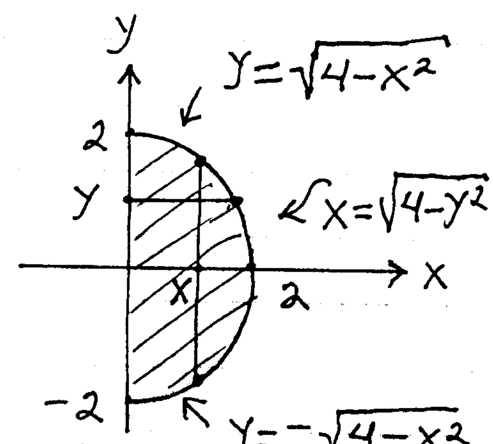
40) $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$

$= \int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx$



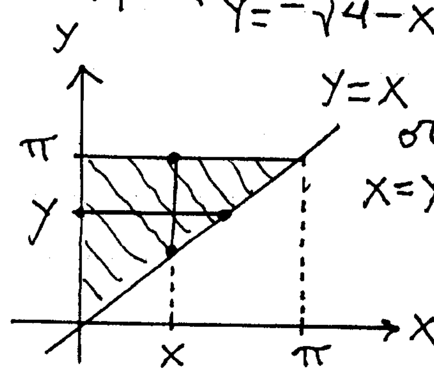
42) $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\, dy\, dx$

$= \int_{-2}^2 \int_0^{\sqrt{4-y^2}} 6x\, dx\, dy$



47) $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$

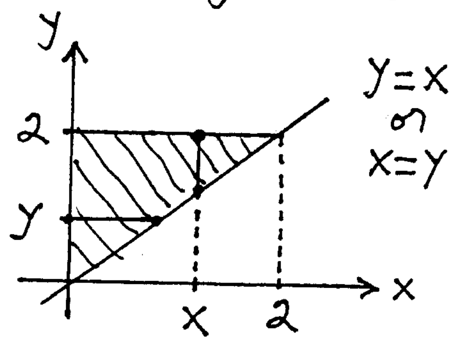
$= \int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy$



$$= \int_0^{\pi} \left(\frac{\sin y}{y} \cdot x \Big|_{x=0}^{x=y} \right) dy = \int_0^{\pi} (\sin y - 0)\, dy$$

$$= -\cos y \Big|_0^{\pi} = -\cos\pi - -\cos 0 = -(-1) + 1 = 2$$

48) $\int_0^2 \int_x^2 2y^2 \sin xy \, dy\, dx$



$$= \int_0^2 \int_0^y 2y^2 \sin xy \, dx\, dy$$

$$= \int_0^2 \left(2y^2 \cdot \frac{-1}{y} \cos xy \Big|_{x=0}^{x=y} \right) dy$$

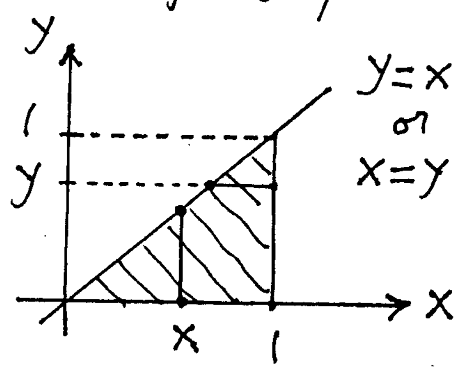
$$= \int_0^2 \left(-2y \cos y^2 - -2y \overset{1}{\cancel{\cos 0}} \right) dy$$

$$= \int_0^2 \left(-2y \cos y^2 + 2y \right) dy$$

$$= \left(-\sin y^2 + y^2 \right) \Big|_0^2 = (-\sin 4 + 4) - (-\overset{0}{\cancel{\sin 0}} + 0)$$

$$= 4 - \sin 4$$

49) $\int_0^1 \int_y^1 x^2 e^{xy} \, dx\, dy$



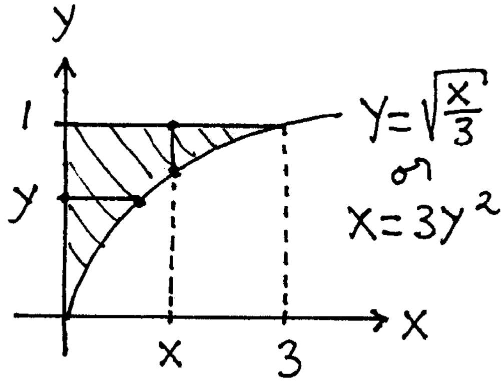
$$= \int_0^1 \int_0^x x^2 e^{xy} \, dy\, dx$$

$$= \int_0^1 \left(x^2 \cdot \frac{1}{x} e^{xy} \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \left(x e^{x^2} - x \cdot \overset{1}{\cancel{e^0}} \right) dx = \left(\tfrac{1}{2} e^{x^2} - \tfrac{1}{2} x^2 \right) \Big|_0^1$$

$$= \left(\tfrac{1}{2} e - \tfrac{1}{2} \right) - \left(\tfrac{1}{2} e^0 - 0 \right) = \tfrac{1}{2} e - 1$$

52) $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\,dy\,dx$

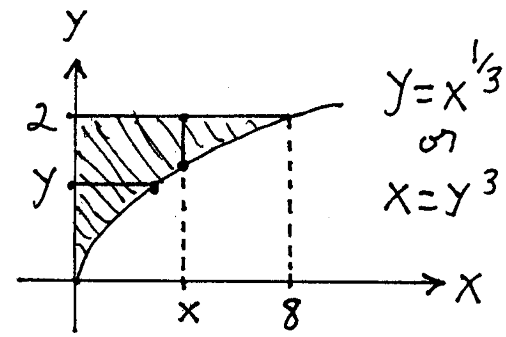


$= \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$

$= \int_0^1 \left(e^{y^3}\cdot x \Big|_{x=0}^{x=3y^2}\right) dy = \int_0^1 3y^2 e^{y^3}\,dy$

$= e^{y^3}\Big|_0^1 = e^1 - e^0 = e-1$

54.) $\int_0^8 \int_{x^{1/3}}^{2} \frac{1}{y^4+1}\,dy\,dx$



$= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$

$= \int_0^2 \left(\frac{1}{y^4+1}\cdot x \Big|_{x=0}^{x=y^3}\right) dy$

$= \int_0^2 \frac{y^3}{y^4+1}\,dy = \frac{1}{4}\ln|y^4+1|\,\Big|_0^2$

$= \frac{1}{4}\ln 17 - \frac{1}{4}\cancel{\ln 1}^{\,0} = \frac{1}{4}\ln 17$

55.) $|x|+|y|=1$ (graph it)

<u>case 1</u>: $x\geq 0,\ y\geq 0 \rightarrow x+y=1 \rightarrow y=1-x$

<u>case 2</u>: $x\leq 0,\ y\geq 0 \rightarrow -x+y=1 \rightarrow y=1+x$

<u>case 3</u>: $x\leq 0,\ y\leq 0 \rightarrow -x+-y=1 \rightarrow y=-x-1$

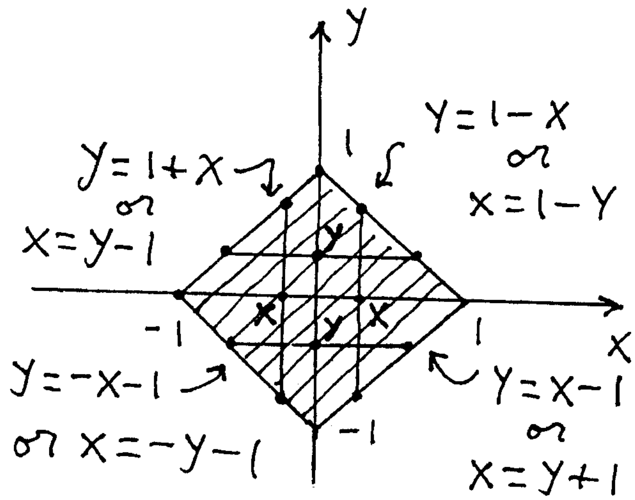
<u>case 4</u>: $x\geq 0,\ y\leq 0 \rightarrow x+-y=1 \rightarrow y=x-1$

$\iint_R (y-2x^2)\,dA = \int_{-1}^{0}\int_{-x-1}^{x+1} (y-2x^2)\,dy\,dx \ +$

$$\int_0^1 \int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$

$$= \int_{-1}^{0} \int_{-y-1}^{y+1} (y-2x^2)\,dx\,dy$$

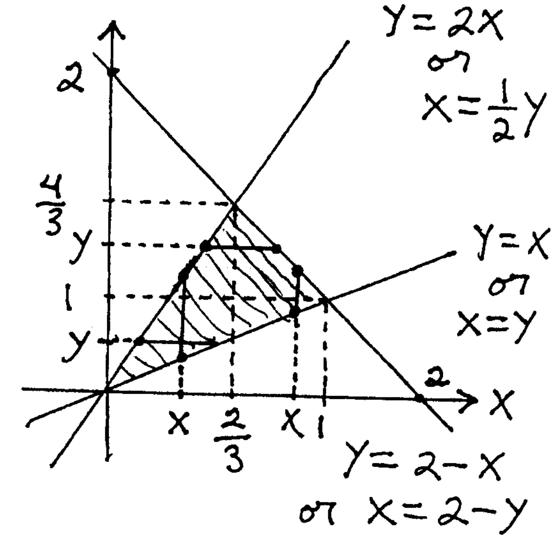
$$+ \int_0^1 \int_{y-1}^{1-y} (y-2x^2)\,dx\,dy$$



56) $\iint_R xy\,dA$

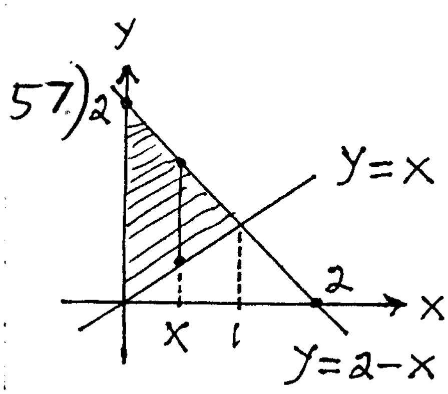
$$= \int_0^{2/3} \int_x^{2x} xy\,dy\,dx$$

$$+ \int_{2/3}^{1} \int_x^{2-x} xy\,dy\,dx$$

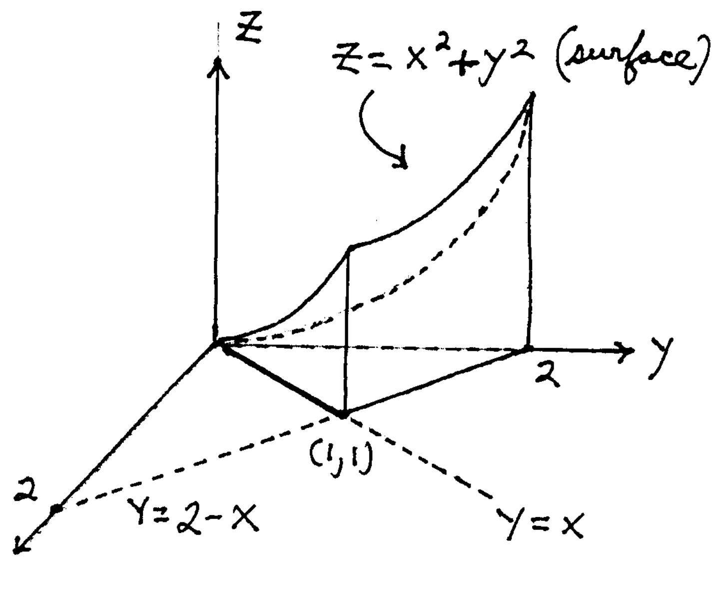


$$= \int_0^1 \int_{\frac{1}{2}y}^{y} xy\,dx\,dy + \int_1^{\frac{4}{3}} \int_{\frac{1}{2}y}^{2-y} xy\,dx\,dy$$

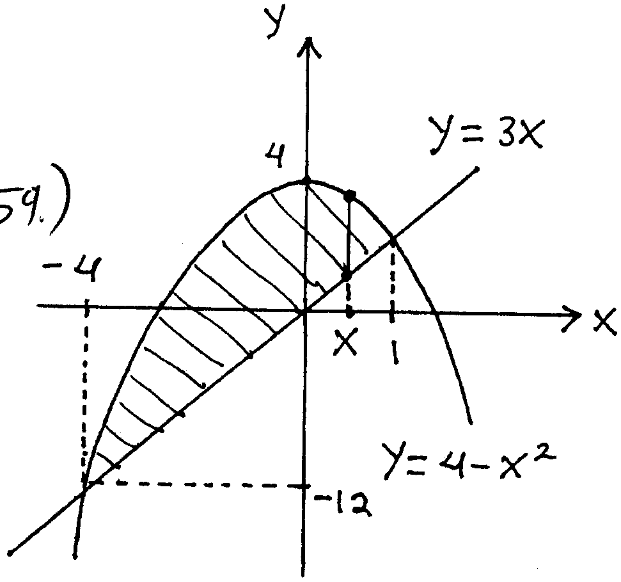
57)



$$\text{Vol} = \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx$$



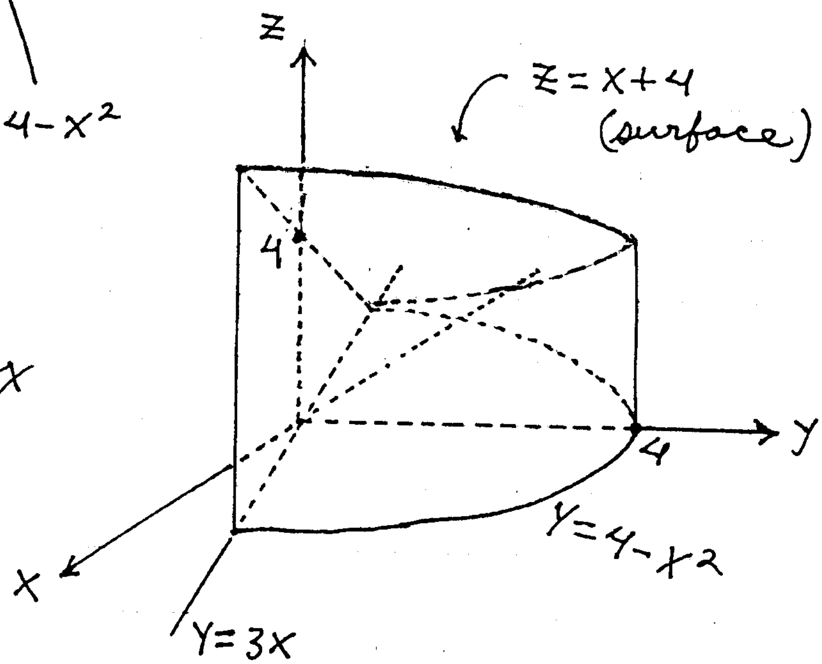
59.)



$$4 - x^2 = 3x \rightarrow x^2 + 3x - 4 = 0$$

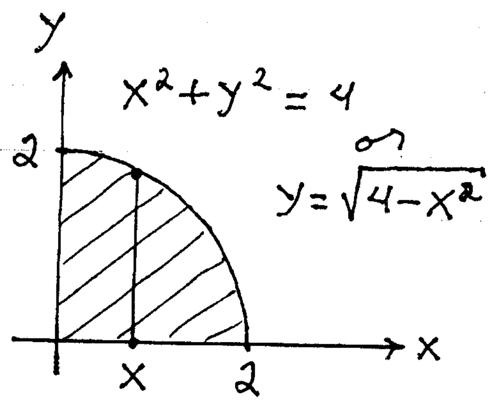
$$\rightarrow (x-1)(x+4) = 0$$

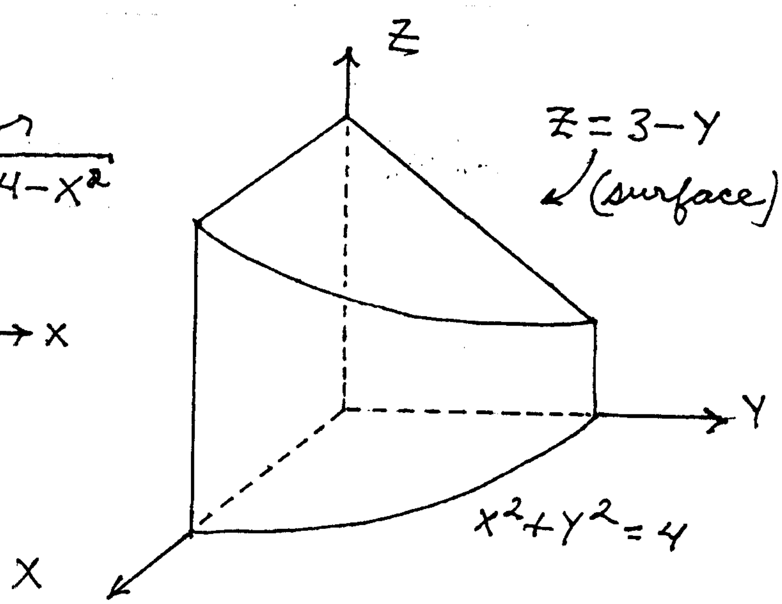
$$x = 1 \qquad x = -4$$



$$Vol = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\, dy\, dx$$

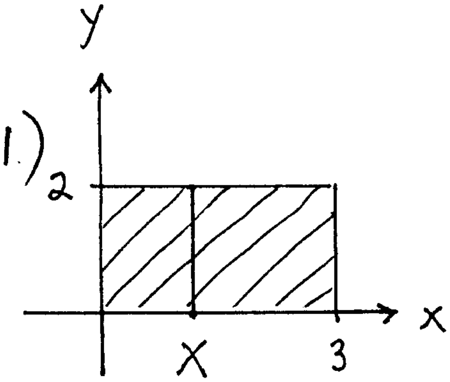
60.)

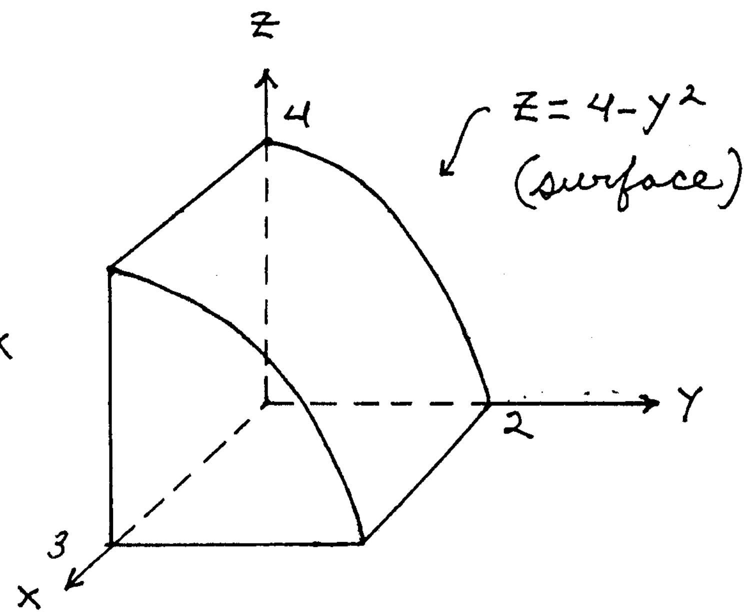




$$Vol = \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} (3-y)\, dy\, dx$$

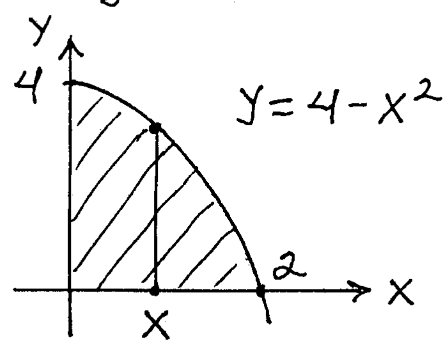
61)

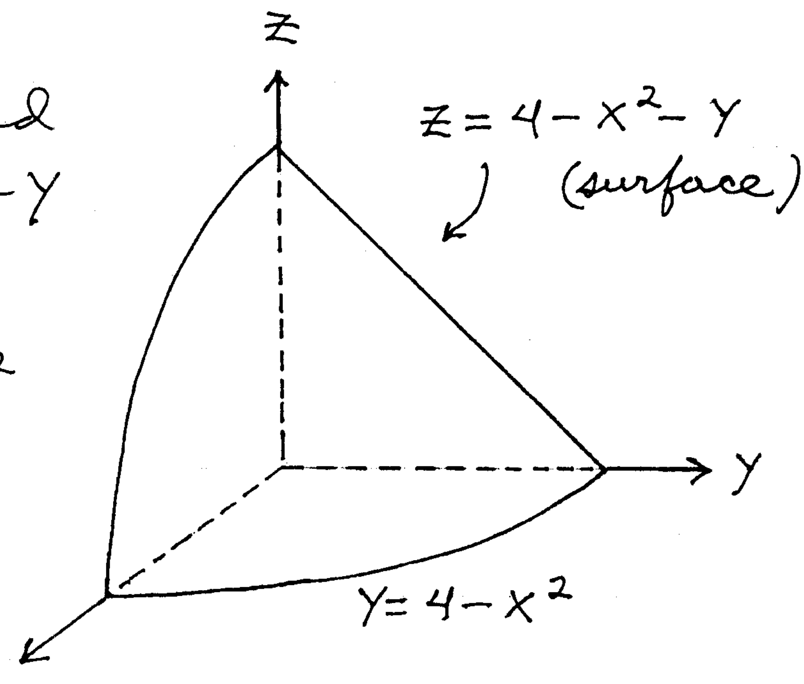




$$\text{Vol} = \int_0^3 \int_0^2 (4-y^2)\,dy\,dx$$

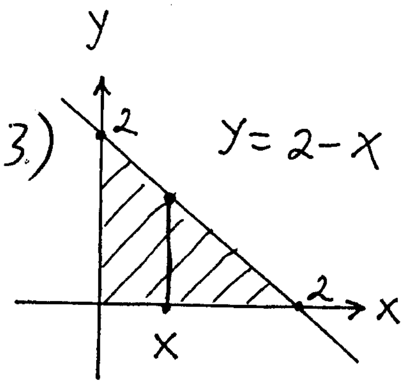
62) $z = 4 - x^2 - y$ and
$z = 0 \rightarrow 0 = 4 - x^2 - y$
$\rightarrow \quad y = 4 - x^2$

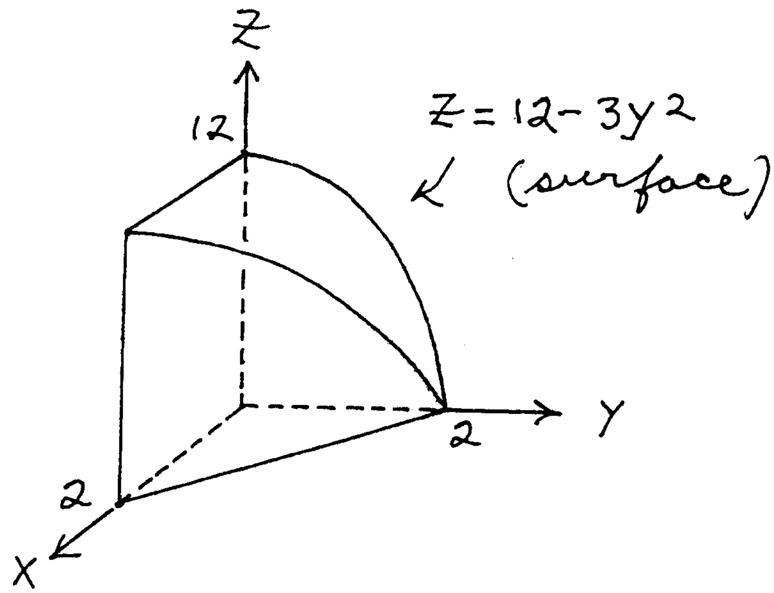




$$\text{Vol} = \int_0^2 \int_0^{4-x^2} (4-x^2-y)\,dy\,dx$$

63.)





$$\text{Vol} = \int_0^2 \int_0^{2-x} (12 - 3y^2)\, dy\, dx$$

69.) $$\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\, dy\, dx = \int_1^\infty \left(\frac{1}{x^3} \ln y \Big|_{y=e^{-x}}^{y=1} \right) dx$$

$$= \int_1^\infty \left(\frac{1}{x^3} \cancel{\ln 1}^{\,0} - \frac{1}{x^3} \ln e^{-x} \right) dx$$

$$= \int_1^\infty \frac{-1}{x^3} \cdot (-x)\, dx = \int_1^\infty \frac{1}{x^2}\, dx$$

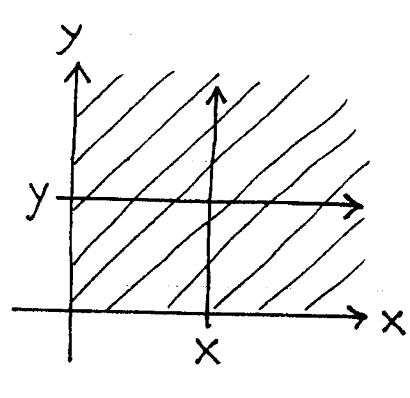
$$= \lim_{A \to \infty} \int_1^A x^{-2}\, dx = \lim_{A \to \infty} -x^{-1} \Big|_1^A$$

$$= \lim_{A \to \infty} \left(\frac{-1}{A} - \frac{-1}{1} \right) = 0 + 1 = 1$$

72.) $$\int_0^\infty \int_0^\infty x e^{-x-2y}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty x e^{-x-2y}\, dy\, dx$$

$$= \int_0^\infty \left(\lim_{A \to \infty} \int_0^A x e^{-x-2y}\, dy \right) dx$$



$$= \int_0^\infty \left(\lim_{A\to\infty} \frac{-1}{2} x e^{-x-2y} \Big|_{y=0}^{y=A} \right) dx$$

$$= \int_0^\infty \left(\lim_{A\to\infty} \left[\frac{-1}{2} \cancel{x e^{-x-2A}}^{\,0} - \frac{-1}{2} x e^{-x} \right] \right) dx$$

$$= \int_0^\infty \frac{1}{2} x e^{-x}\, dx \qquad \left(\text{Let } u = \tfrac{1}{2}x,\ dv = e^{-x}dx \rightarrow du = \tfrac{1}{2}dx,\ v = -e^{-x} \right)$$

$$= \frac{-1}{2} x e^{-x} \Big|_0^\infty - \frac{-1}{2} \int_0^\infty e^{-x}\, dx$$

$$= \lim_{A\to\infty} \left[\frac{-1}{2} x e^{-x} \Big|_0^A + \frac{1}{2} \int_0^A e^{-x}\, dx \right]$$

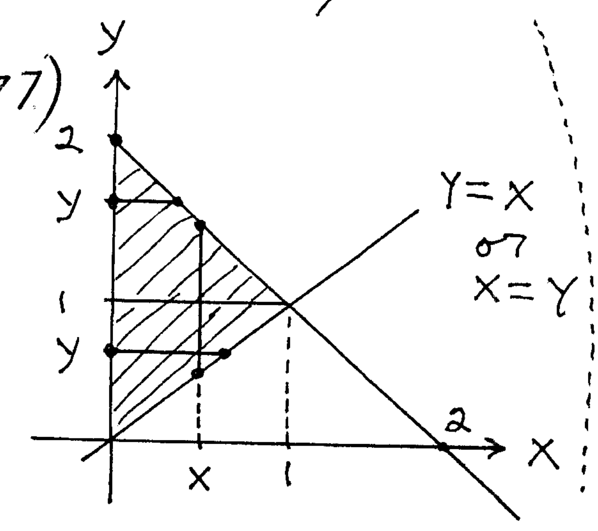
$$= \lim_{A\to\infty} \left[\left(\frac{-1}{2} \cdot \frac{A}{e^A} - 0 \right) + \frac{1}{2} \cdot -e^{-x} \Big|_0^A \right]$$

$$= \lim_{A\to\infty} \left[\frac{-1}{2} \cdot \frac{A}{e^A} + \left(\frac{-1}{2} \cdot \cancel{\frac{1}{e^A}}^{\,0} - \frac{-1}{2}(1) \right) \right]$$

← L'Hopital's Rule, "$\frac{\infty}{\infty}$"

$$= \lim_{A\to\infty} \left[\frac{-1}{2} \cdot \cancel{\frac{1}{e^A}}^{\,0} + \frac{1}{2} \right] = \frac{1}{2}$$

77)



Y=2−X or X=2−Y

$$\int_0^1 \int_0^y (x^2+y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\, dx\, dy$$

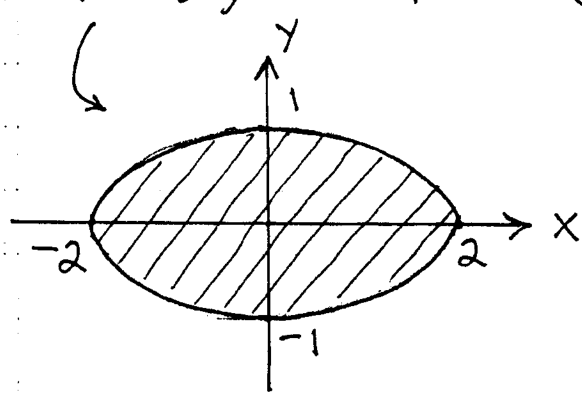
$$= \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx$$

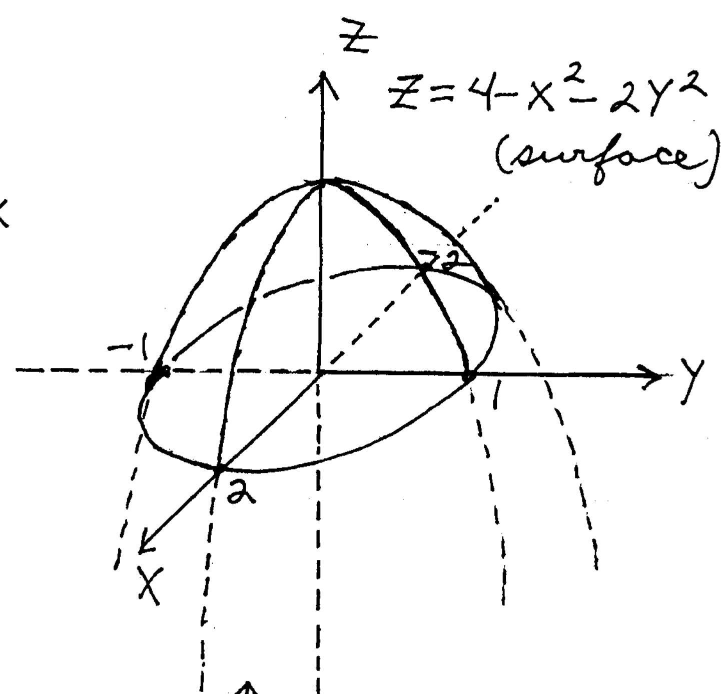
79) Consider trace of surface

$z = 4 - x^2 - 2y^2$ in xy-plane :

$z = 0 \rightarrow 0 = 4 - x^2 - 2y^2 \rightarrow$

$x^2 + 2y^2 = 4$ (ellipse)





i.) If region R is inside ellipse, then solid will be smaller than

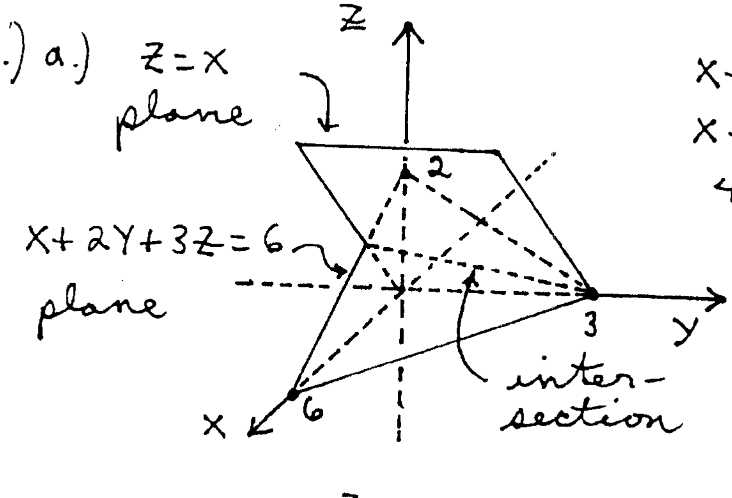
ii.) If region R is outside ellipse, then value of z is NEGATIVE and surface is below xy-plane creating "NEGATIVE" volume

iii) Volume is MAXIMUM when R : pts. (x,y) satisfying

$x^2 + 2y^2 \leq 4$

Worksheet 1 Solutions

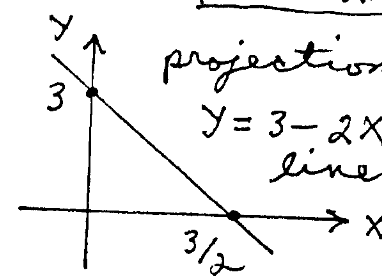
1.) a.)



$x + 2y + 3z = 6$ and $z = x \rightarrow$

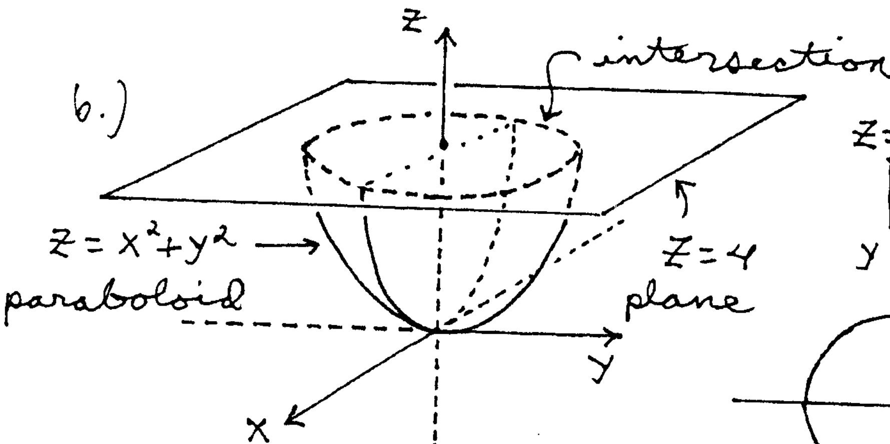
$x + 2y + 3(x) = 6 \rightarrow$

$4x + 2y = 6 \rightarrow \boxed{y = 3 - 2x}$



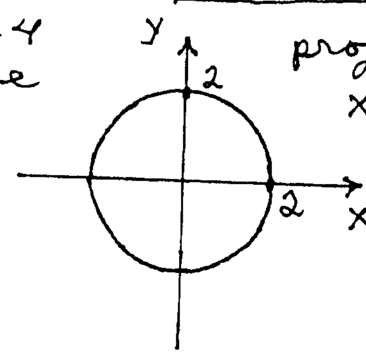
projection: $y = 3 - 2x$ line

b.)



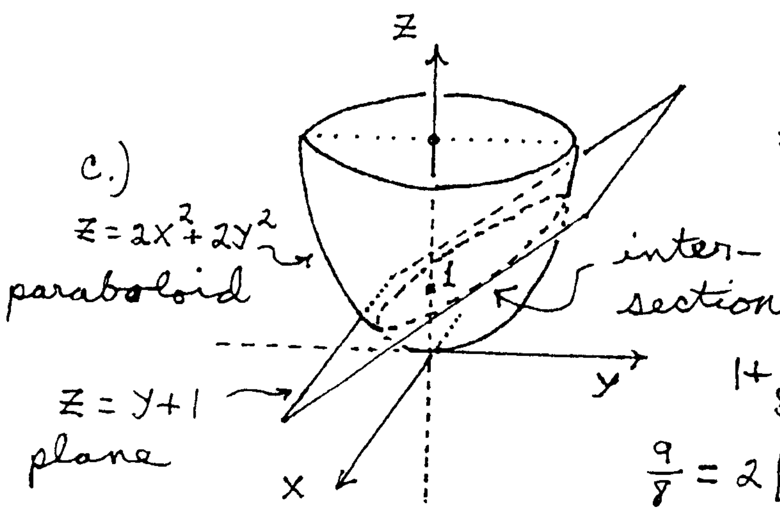
$z = x^2 + y^2$ and $z = 4 \rightarrow$

$\boxed{x^2 + y^2 = 2^2}$



projection: $x^2 + y^2 = 4$ circle

c.)



$z = 2x^2 + 2y^2$ and $z = y + 1 \rightarrow$

$y + 1 = 2x^2 + 2y^2 \rightarrow$

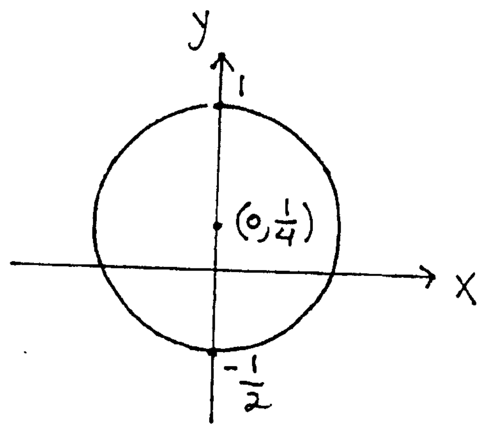
$1 = 2x^2 + 2y^2 - y \rightarrow$

$1 = 2x^2 + 2\left(y^2 - \frac{1}{2}y\right) \rightarrow$

$1 + \frac{1}{8} = 2x^2 + 2\left(y^2 - \frac{1}{2}y + \frac{1}{16}\right) \rightarrow$

$\frac{9}{8} = 2\left[x^2 + \left(y - \frac{1}{4}\right)^2\right] \rightarrow$

$$\boxed{x^2 + \left(y - \frac{1}{4}\right)^2 = \left(\frac{3}{4}\right)^2}$$

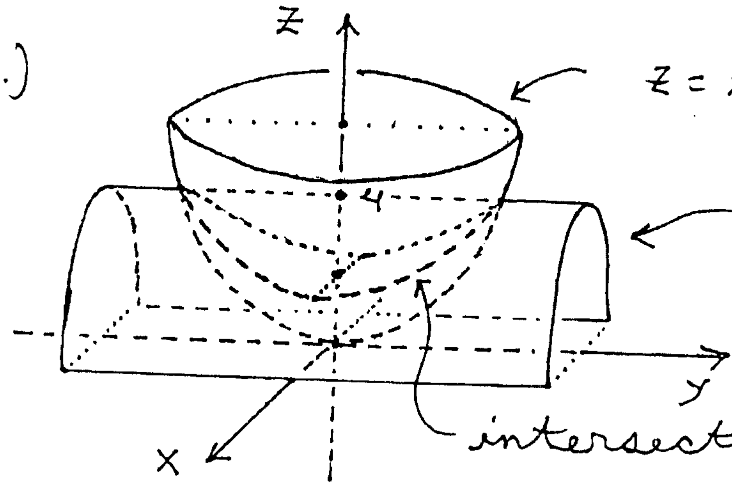


projection:

$$x^2 + \left(y - \frac{1}{4}\right)^2 = \left(\frac{3}{4}\right)^2$$

circle

d.)



$z = x^2 + y^2$ paraboloid,

$z = 4 - 3x^2$ parabolic cylinder

$z = x^2 + y^2$ and $z = 4 - 3x^2 \rightarrow$

$4 - 3x^2 = x^2 + y^2 \rightarrow$

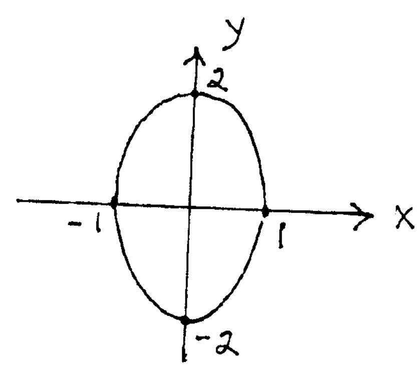
$4 = 4x^2 + y^2 \rightarrow$

$$\boxed{1 = x^2 + \frac{y^2}{2^2}}$$

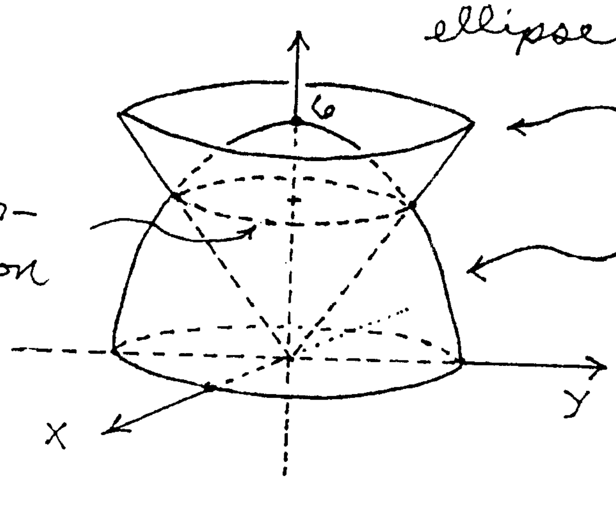
projection:

$$1 = x^2 + \frac{y^2}{4}$$

ellipse



e.)



inter-section

$z = \sqrt{x^2 + y^2}$ cone,

$z = 6 - x^2 - y^2$ paraboloid

$z = \sqrt{x^2+y^2}$ and $z = 6 - x^2 - y^2 \rightarrow$

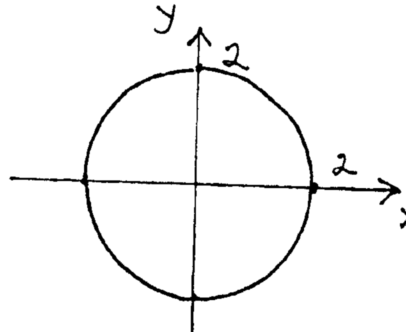
$z^2 = x^2+y^2$ and $z = 6 - (x^2+y^2) \rightarrow$

$z = 6 - z^2 \rightarrow$

$z^2 + z - 6 = 0 \rightarrow (z-2)(z+3) = 0 \rightarrow$

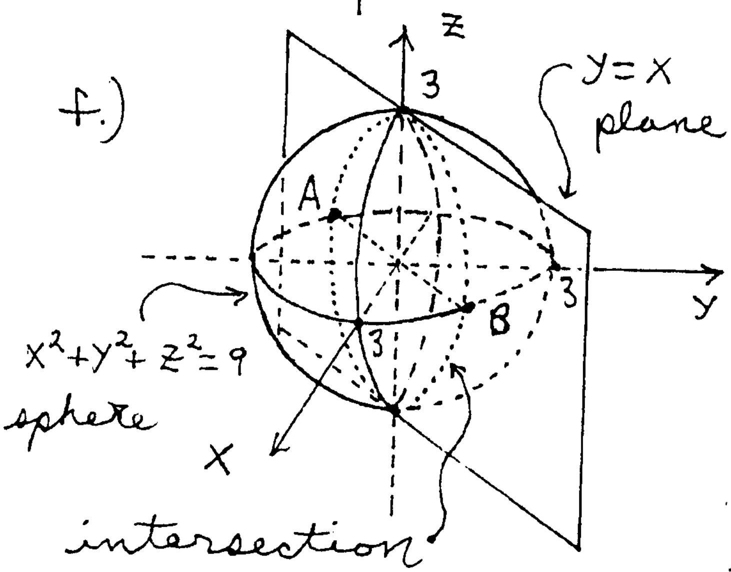
$z = -3$ (NO) or $\underline{z=2}$; then

$2 = \sqrt{x^2+y^2} \rightarrow \boxed{x^2+y^2 = 2^2}$;

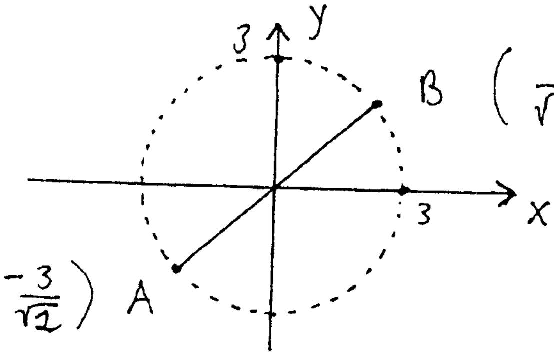


projection : $x^2+y^2 = 4$ circle

f.)



Since the plane $y=x$ is perpendicular to the xy-plane, the projection of the intersection in the xy-plane is the <u>line segment joining</u> <u>A and B</u>.

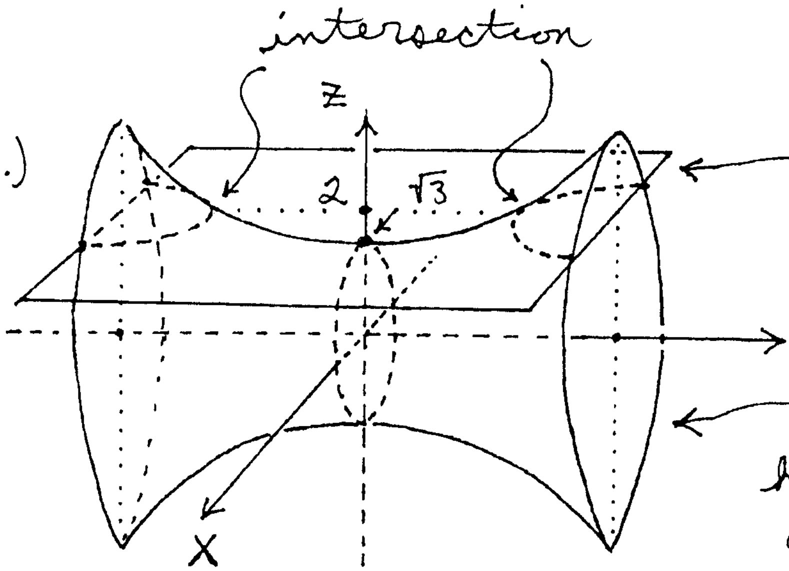


B $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$ A

projection : segment AB

g.)



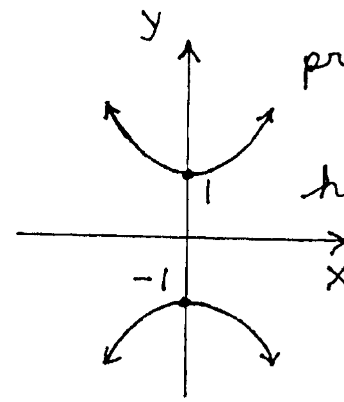
$\leftarrow z = 2$ plane,

$\leftarrow x^2 + z^2 = y^2 + 3$ hyperboloid of one sheet

$x^2 + z^2 = y^2 + 3$ and $z = 2 \rightarrow$

$x^2 + 4 = y^2 + 3 \rightarrow$

$\boxed{y^2 - x^2 = 1}$

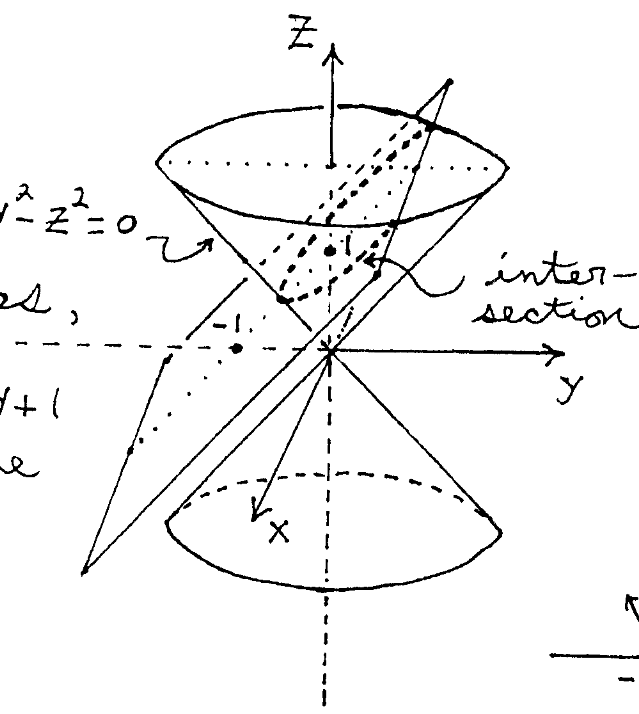


projection: $y^2 - x^2 = 1$ hyperbola

h.)

$x^2 + y^2 - z^2 = 0$ cones,

$z = y + 1$ plane



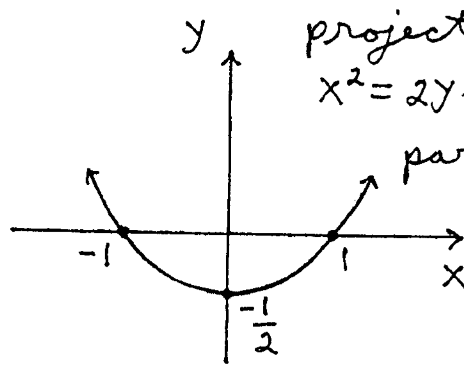
inter-section

$x^2 + y^2 - z^2 = 0$ and $z = y + 1 \rightarrow$

$x^2 + y^2 - (y+1)^2 = 0 \rightarrow$

$x^2 + \cancel{y^2} - \cancel{y^2} - 2y - 1 = 0 \rightarrow$

$\boxed{x^2 = 2y + 1}$



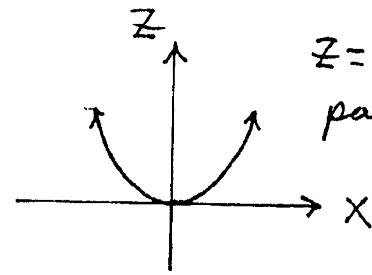
projection: $x^2 = 2y + 1$ parabola

2.) $z = x^2 + y - 1$ and $y = x^2 + 1$

a.) xz-plane projection:

$z = x^2 + (x^2+1) - 1 \rightarrow$

$\boxed{z = 2x^2}$



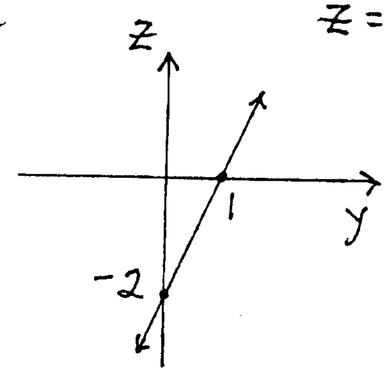
$z = 2x^2$
parabola

b.) yz-plane projection:

$y = x^2 + 1 \rightarrow x^2 = y - 1$ so

$z = (y-1) + y - 1 \rightarrow$

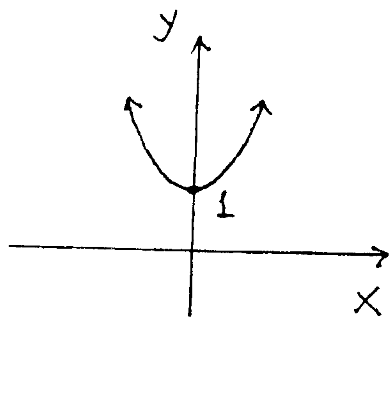
$\boxed{z = 2y - 2}$



$z = 2y - 2$
line

c.) xy-plane projection:

Since $y = x^2 + 1$ is a cylinder (surface) perpendicular to the xy-plane, the projection of the intersection in the xy-plane is $\boxed{y = x^2 + 1}$.



$y = x^2 + 1$
parabola