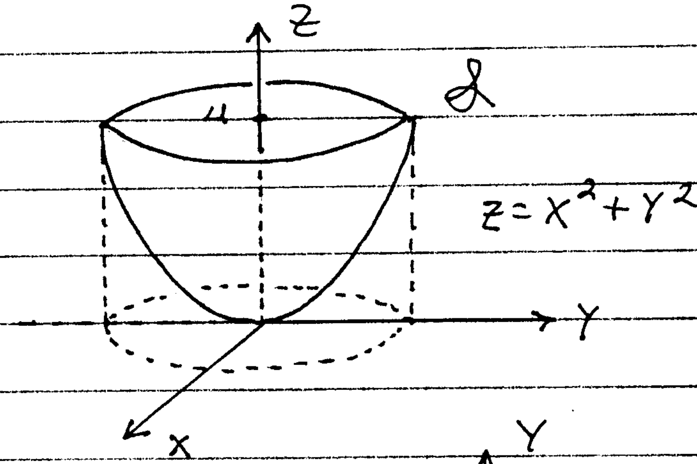
# Section 16.5

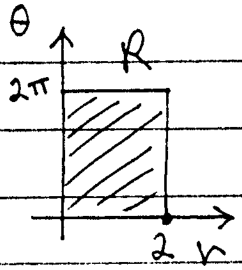
1.) $z = x^2 + y^2$ and $z \le 4 \rightarrow$

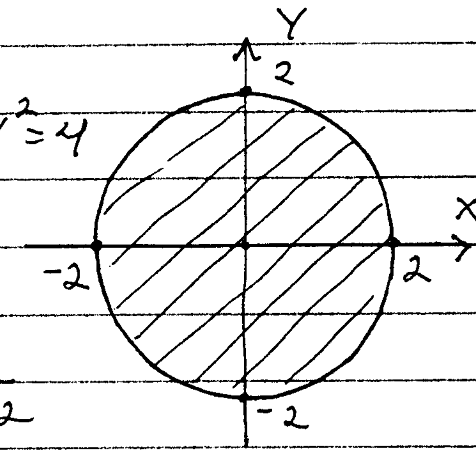
$4 = x^2 + y^2$



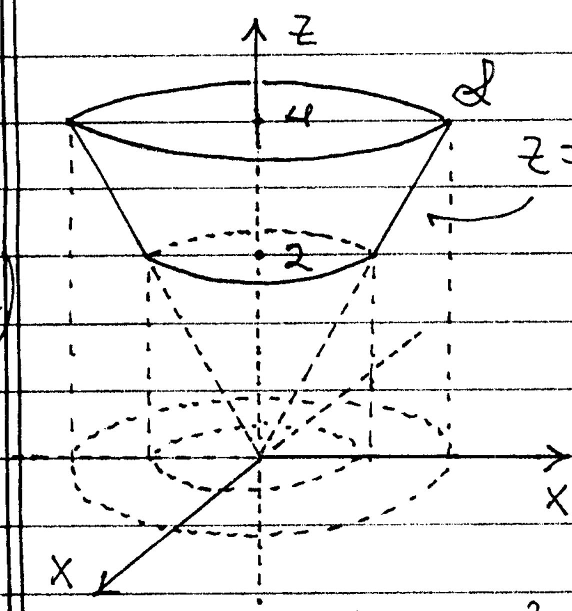

$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = x^2+y^2 = r^2 \end{cases}$$

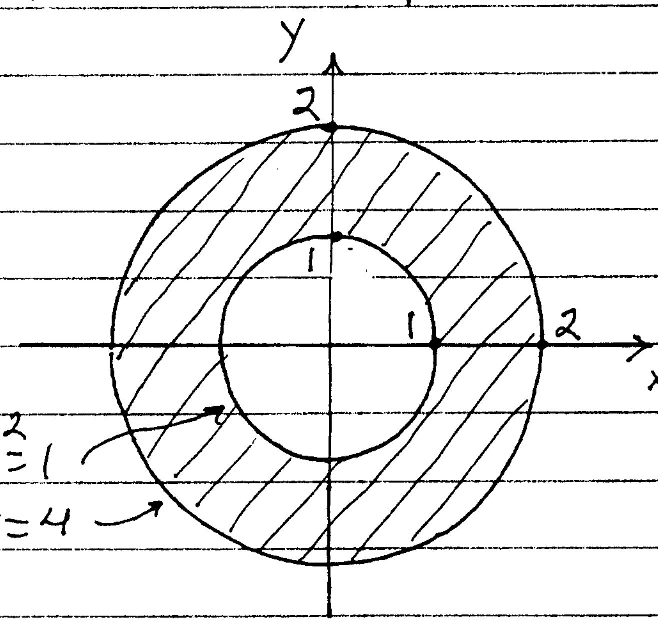
for $0 \le \theta \le 2\pi$, $0 \le r \le 2$.




4.)



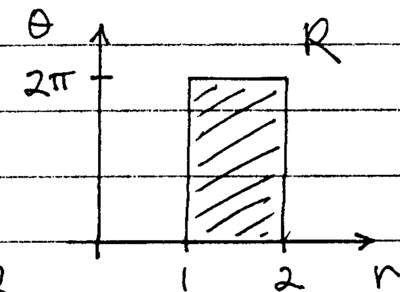



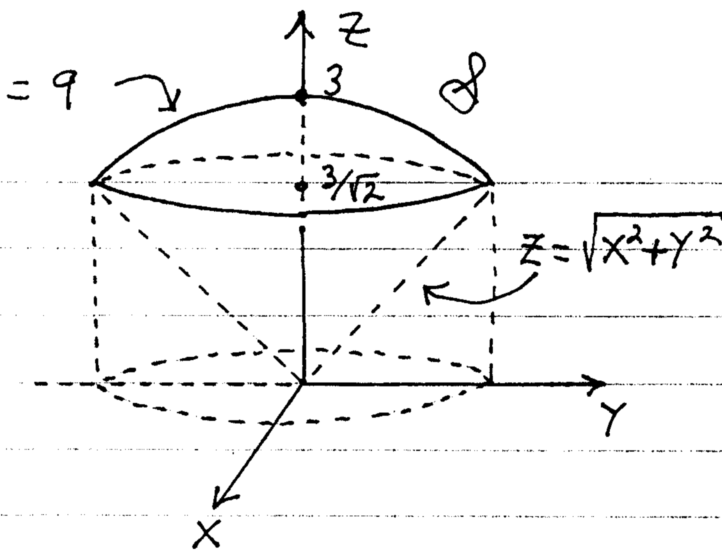

$2 = 2\sqrt{x^2+y^2} \rightarrow x^2+y^2=1$

$4 = 2\sqrt{x^2+y^2} \rightarrow x^2+y^2=4$

$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 2\sqrt{x^2+y^2} = 2\sqrt{r^2} = 2r \end{cases}$$

for $0 \le \theta \le 2\pi$, $1 \le r \le 2$

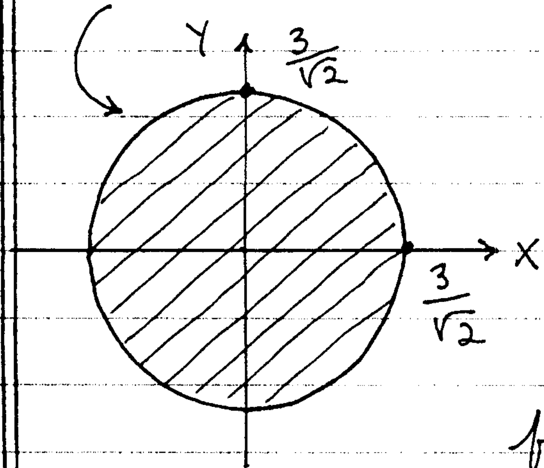


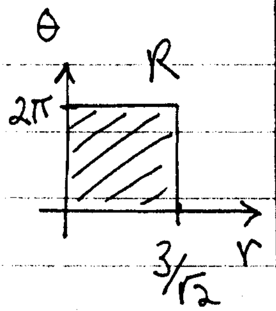



5.) $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 9$

$\rightarrow 2x^2+2y^2 = 9$

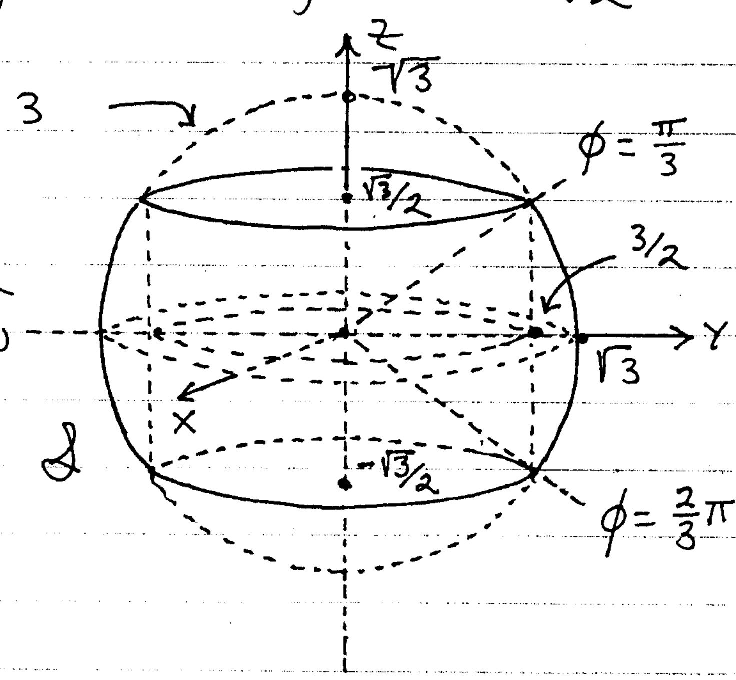
$\rightarrow x^2+y^2 = 9/2$




$$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{9-(x^2+y^2)} = \sqrt{9-r^2} \end{cases}$$

for $0 \le \theta \le 2\pi$, $0 \le r \le 3/\sqrt{2}$




7.) $z = \pm\sqrt{3}/2 \rightarrow$

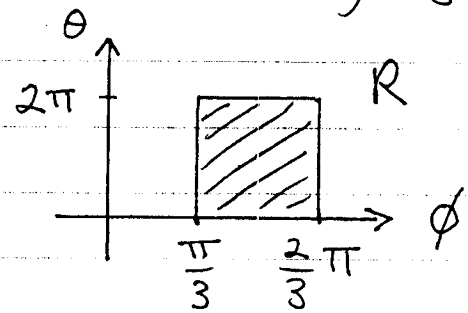
$x^2+y^2+\frac{3}{4} = 3 \rightarrow$

$x^2+y^2 = 9/4 = (3/2)^2$,

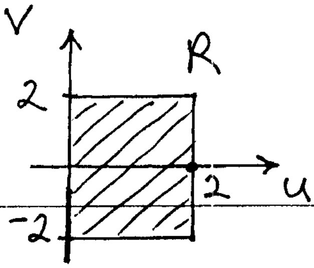
use spherical coordinates: $S$

$$S: \begin{cases} x = \sqrt{3}\sin\phi\cos\theta \\ y = \sqrt{3}\sin\phi\sin\theta \\ z = \sqrt{3}\cos\phi \end{cases}$$

for $0 \le \theta \le 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{2}{3}\pi$

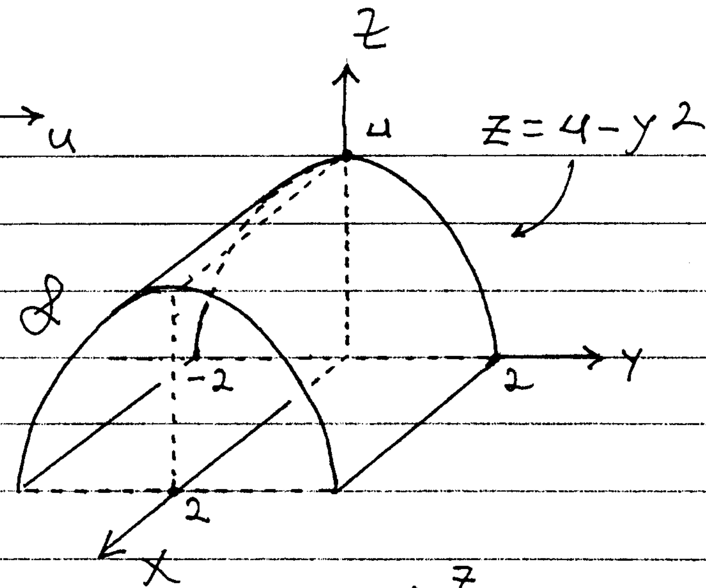


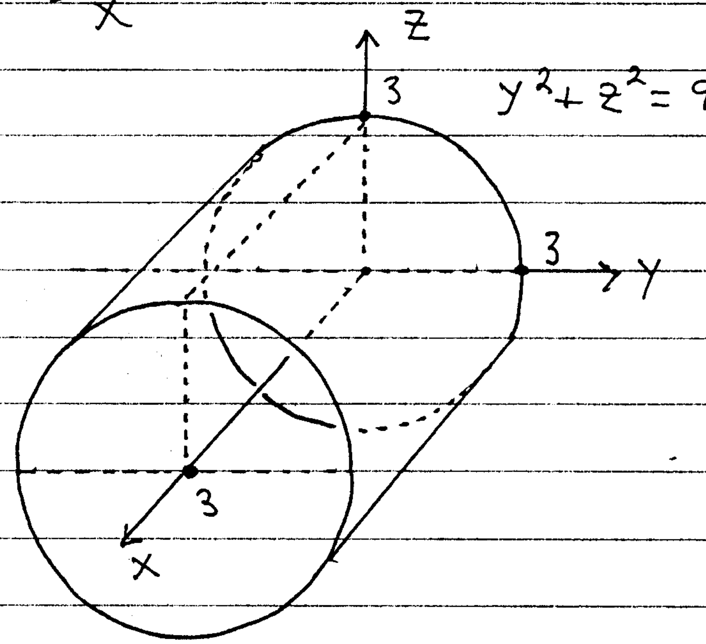
9.)




$$S: \begin{cases} x = u \\ y = v \\ z = 4 - v^2 \end{cases}$$

for $0 \le u \le 2$,
$-2 \le v \le 2$

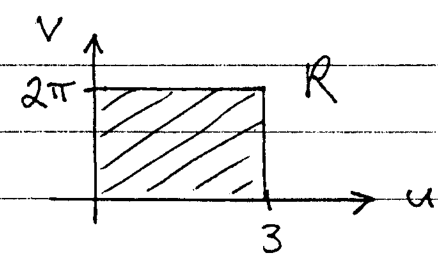


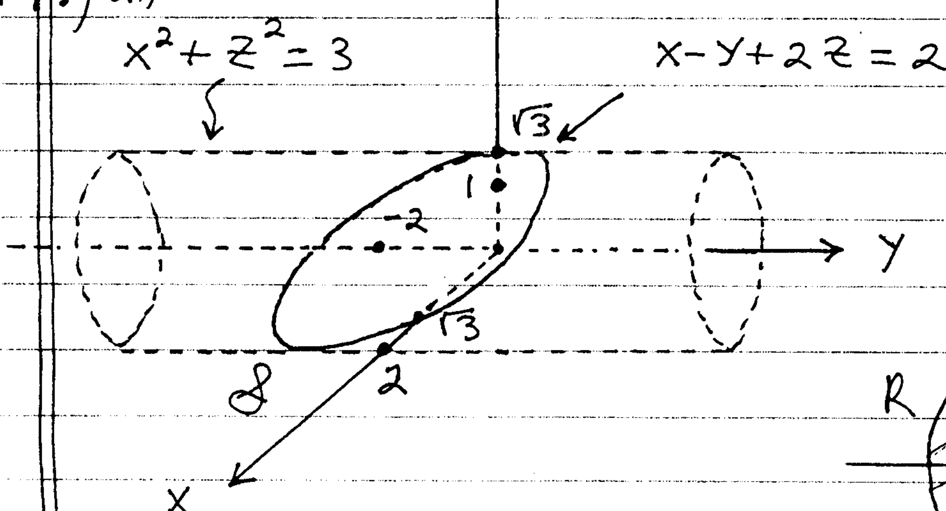

11.)

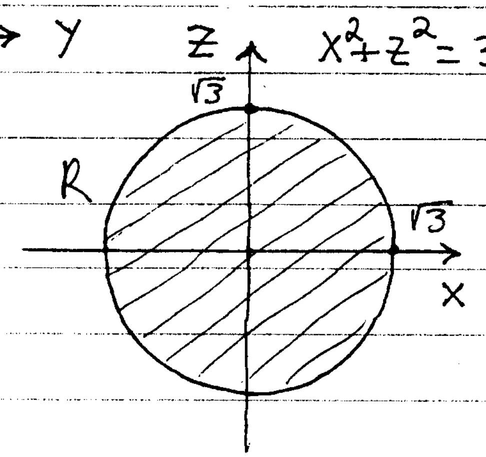
$$S: \begin{cases} x = u \\ y = 3\cos v \\ z = 3\sin v \end{cases}$$

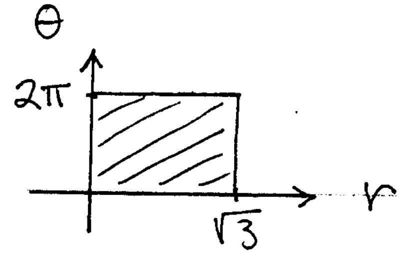
for $0 \le u \le 3$,
$0 \le v \le 2\pi$




14.) a.)



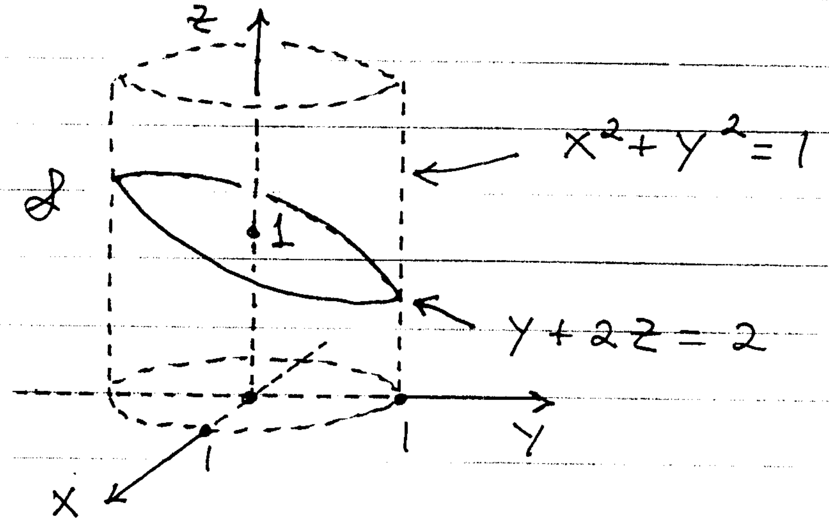



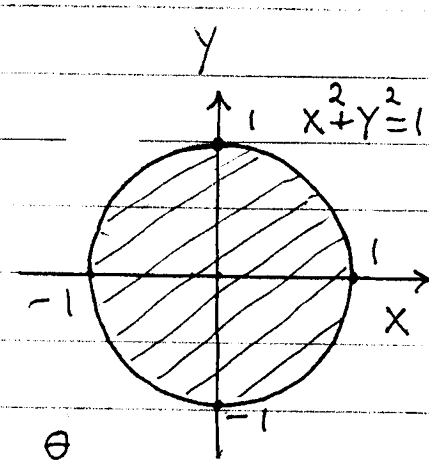


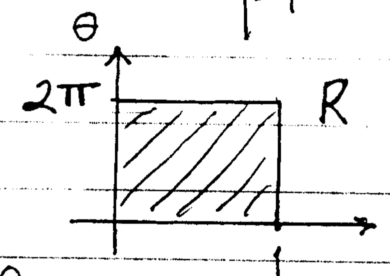

$$S : \begin{cases} x = r\cos\theta \\ z = r\sin\theta \\ y = x + 2z - 2 = r\cos\theta + 2r\sin\theta - 2 \end{cases}$$

for $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$

17.)




$$S : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - \frac{1}{2}y = 1 - \frac{1}{2}r\sin\theta \end{cases}$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi$ ; then

$$\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + (\tfrac{-1}{2}r\cos\theta)\vec{k},$$

$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + (\tfrac{-1}{2}\sin\theta)\vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & \frac{-1}{2}r\cos\theta \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \end{vmatrix}$$

$$= (\tfrac{-1}{2}r\sin\theta\cos\theta - \tfrac{-1}{2}r\sin\theta\cos\theta)\vec{i}$$

$$- (\tfrac{1}{2}r\sin^2\theta - \tfrac{-1}{2}r\cos^2\theta)\vec{j}$$

$+ \left(-r\sin^2\theta - r\cos^2\theta\right)\vec{k}$

$= (0)\vec{i} - \frac{1}{2}r\,(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\vec{j} - r\,(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\vec{k}$

$= (0)\vec{i} + \left(-\frac{1}{2}r\right)\vec{j} + (-r)\vec{k}$ , and

$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{\left(-\frac{1}{2}r\right)^2 + (-r)^2} = \sqrt{\frac{1}{4}r^2 + r^2}$

$= \frac{\sqrt{5}}{2} r$ ; then

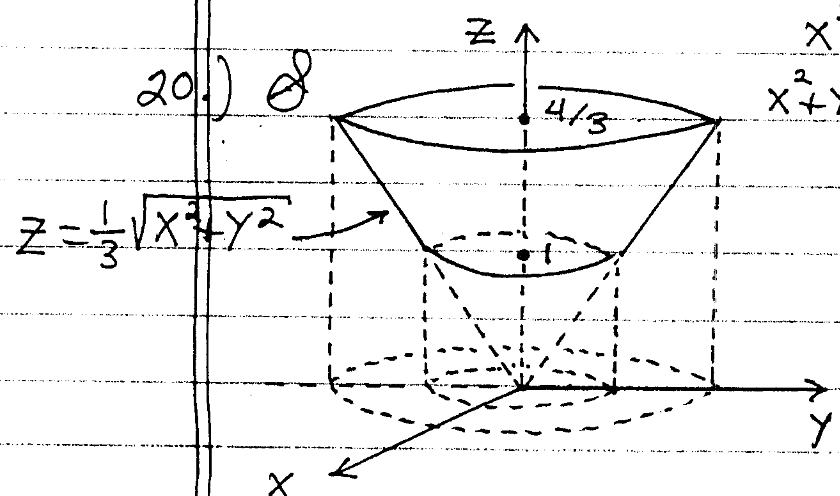
$\text{Area } S = \iint_S 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_r|\, dA$

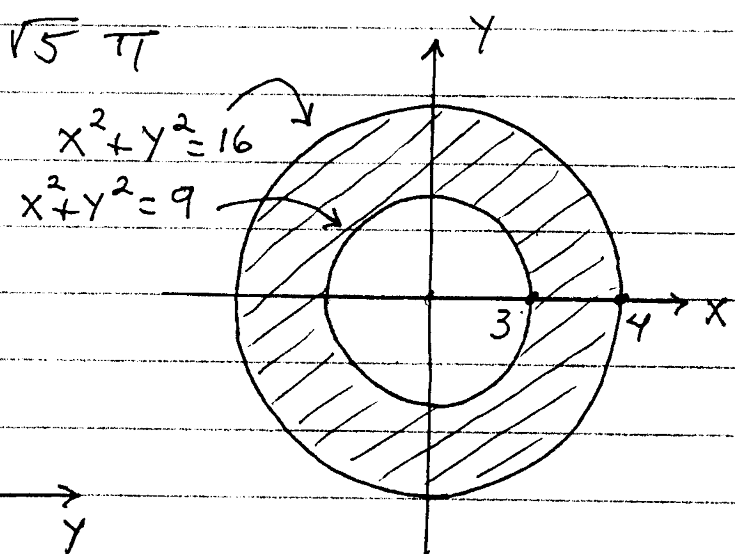
$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2} r\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{\sqrt{5}}{2}\cdot \frac{1}{2} r^2 \Big|_{r=0}^{r=1}\right) d\theta = \int_0^{2\pi} \frac{1}{4}\sqrt{5}\, d\theta$

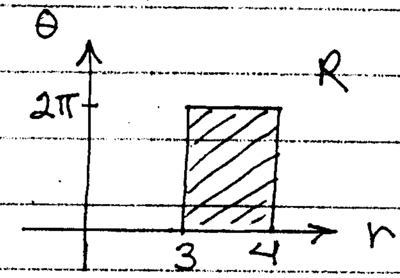
$= \frac{1}{4}\sqrt{5}\cdot 2\pi = \frac{1}{2}\sqrt{5}\,\pi$

20.) $S$



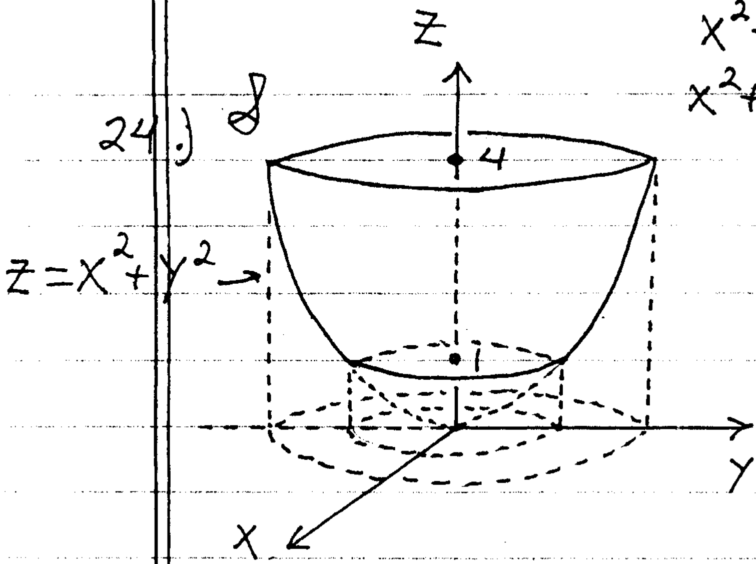



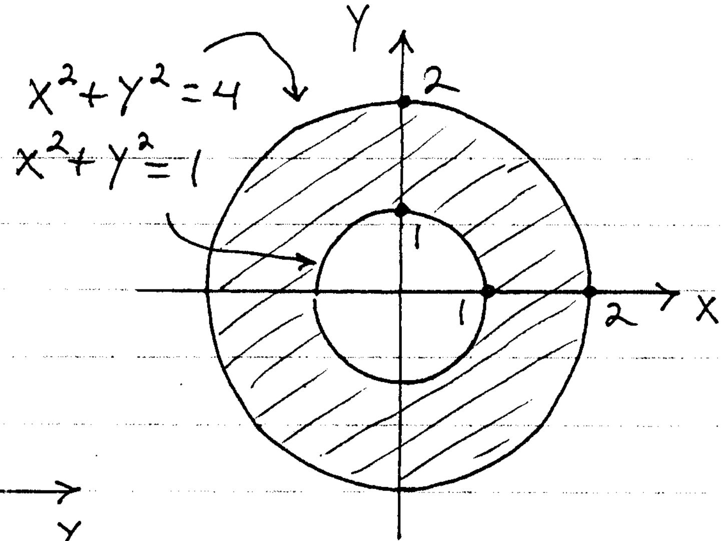

$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \frac{1}{3}\sqrt{r^2} = \frac{1}{3}r \end{cases}$ for $3 \le r \le 4$, $0 \le \theta \le 2\pi$



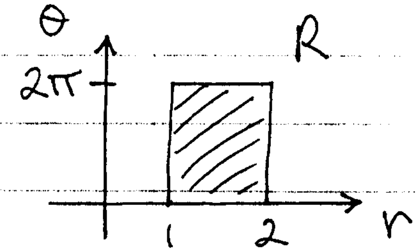
$$\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + (0)\vec{k},$$
$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + \left(\tfrac{1}{3}\right)\vec{k}, \text{ then}$$
$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1/3 \end{vmatrix}$$
$$= \left(\tfrac{1}{3}r\cos\theta - 0\right)\vec{i} - \left(-\tfrac{1}{3}r\sin\theta - 0\right)\vec{j} + \left(-r\sin^2\theta - r\cos^2\theta\right)\vec{k}$$
$$= \left(\tfrac{1}{3}r\cos\theta\right)\vec{i} + \left(\tfrac{1}{3}r\sin\theta\right)\vec{j} - r(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\vec{k}$$
$$= \left(\tfrac{1}{3}r\cos\theta\right)\vec{i} + \left(\tfrac{1}{3}r\sin\theta\right)\vec{j} + (-r)\vec{k};$$
$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{\left(\tfrac{1}{3}r\cos\theta\right)^2 + \left(\tfrac{1}{3}r\sin\theta\right)^2 + (-r)^2}$$
$$= \sqrt{\tfrac{1}{9}r^2\cos^2\theta + \tfrac{1}{9}r^2\sin^2\theta + r^2}$$
$$= \sqrt{\tfrac{1}{9}r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2}$$
$$= \sqrt{\tfrac{10}{9}r^2} = \frac{\sqrt{10}}{3}r \text{ ; then}$$
$$\text{Area } S = \iint_S 1\,dS = \iint_R |\vec{r}_\theta \times \vec{r}_r|\,dA$$
$$= \int_0^{2\pi}\int_3^4 \frac{\sqrt{10}}{3} r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{\sqrt{10}}{3}\cdot\frac{1}{2}r^2\Big|_{r=3}^{r=4}\right)d\theta$$
$$= \int_0^{2\pi}\left(\frac{16}{6}\sqrt{10} - \frac{9}{6}\sqrt{10}\right)d\theta = \frac{7}{6}\sqrt{10}\,\theta\Big|_0^{2\pi}$$
$$= \frac{7}{3}\sqrt{10}\,\pi$$

24.)




$$S: \begin{cases} x = r\cos\theta & \text{for } 1 \le r \le 2, \\ y = r\sin\theta & 0 \le \theta \le 2\pi \\ z = r^2 & \end{cases}$$




$\vec{r}_\theta = (-r\sin\theta)\,\vec{i} + (r\cos\theta)\,\vec{j} + (0)\,\vec{k}$,

$\vec{r}_r = (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + (2r)\,\vec{k}$, then

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}$$

$$= (2r^2\cos\theta - 0)\,\vec{i} - (-2r^2\sin\theta - 0)\,\vec{j} + (-r\sin^2\theta - r\cos^2\theta)\,\vec{k}$$

$$= (2r^2\cos\theta)\,\vec{i} + (2r^2\sin\theta)\,\vec{j} - r\,(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\,\vec{k}$$

$$= (2r^2\cos\theta)\,\vec{i} + (2r^2\sin\theta)\,\vec{j} + (-r)\,\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(2r^2\cos\theta)^2 + (2r^2\sin\theta)^2 + (-r)^2}$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2}$$

$= \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$ ; then

$\text{Area } S = \iint_S 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_\phi|\, dA$

$= \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{8}\cdot\frac{2}{3}(4r^2+1)^{3/2}\Big|_{r=1}^{r=2}\right) d\theta$

$= \int_0^{2\pi} \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}\right) d\theta$

$= \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}\right)\cdot \theta \Big|_0^{2\pi}$

$= \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}(2\pi)\right)$

$= \frac{1}{6}\pi\left((17)^{3/2} - (5)^{3/2}\right)$.

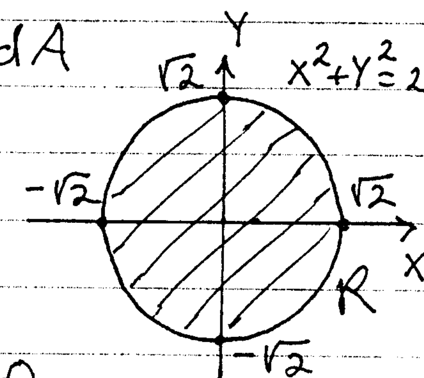
37.) $\mathcal{S}: x^2+y^2-z=0$ and $z=2 \rightarrow$

$R: x^2+y^2=2$ ; then

$\vec{\nabla} f = (2x)\,\vec{i} + (2y)\,\vec{j} + (-1)\,\vec{k}$ so that

$$\sec\gamma = \frac{\sqrt{(2x)^2+(2y)^2+(-1)^2}}{|-1|} = \sqrt{4x^2+4y^2+1} ;$$

$$\text{Area} = \iint_{\mathcal{S}} 1\, dS = \iint_R \sec\gamma \cdot dA$$




$$= \iint_R \sqrt{4x^2+4y^2+1}\; dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\cdot\frac{1}{8}\,(4r^2+1)^{3/2}\Big|_{r=0}^{r=\sqrt{2}}$$

$$= \int_0^{2\pi}\left(\frac{1}{12}(9)^{3/2} - \frac{1}{12}(1)^{3/2}\right) d\theta$$

$$= \int_0^{2\pi}\left(\frac{27}{12}-\frac{1}{12}\right) d\theta = \int_0^{2\pi}\frac{26}{12}\, d\theta$$

$$= \int_0^{2\pi}\frac{13}{6}\, d\theta = \frac{13}{6}\,\theta\Big|_0^{2\pi} = \frac{13}{6}\cdot 2\pi = \frac{13}{3}\pi$$

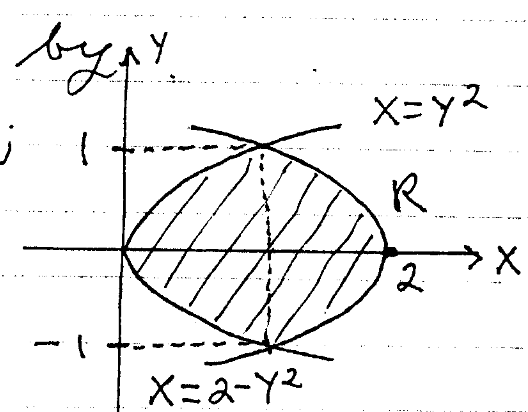
39.) $\mathcal{S}: x+2y+2z=5$ cut by

$x=y^2$ and $x=2-y^2$ ;

then

$\vec{\nabla} f = (1)\,\vec{i} + (2)\,\vec{j} + (2)\,\vec{k}$

so that



$\sec \gamma = \dfrac{\sqrt{(1)^2+(2)^2+(2)^2}}{|2|} = \dfrac{3}{2}$ ;

$\text{Area} = \iint_S 1\, dS = \iint_R \sec\gamma \cdot dA$

$= \iint_R \frac{3}{2}\, dA = \int_{-1}^{1}\int_{y^2}^{2-y^2} \frac{3}{2}\, dx\, dy$

$= \int_{-1}^{1} \left( \frac{3}{2} x \Big|_{x=y^2}^{x=2-y^2} \right) dy$

$= \int_{-1}^{1} \left( \frac{3}{2}(2-y^2) - \frac{3}{2} y^2 \right) dy$

$= \int_{-1}^{1} \left( 3 - \frac{3}{2} y^2 - \frac{3}{2} y^2 \right) dy$

$= \int_{-1}^{1} (3 - 3y^2)\, dy = (3y - y^3) \Big|_{-1}^{1}$

$= (3-1) - (-3+1) = 2 - (-2) = 4$

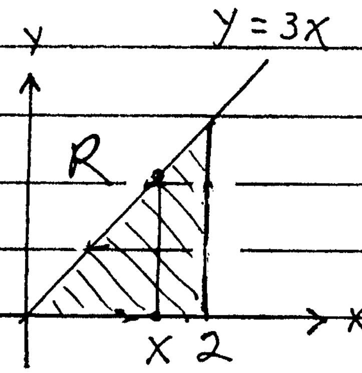
41.) $S: x^2 - 2y - 2z = 0$ and

$\vec{\nabla} f = (2x)\vec{i} + (-2)\vec{j} + (-2)\vec{k}$,

so that

$\sec\gamma = \dfrac{\sqrt{(2x)^2+(-2)^2+(-2)^2}}{|-2|}$

$= \frac{1}{2}\sqrt{4(x^2+2)} = \sqrt{x^2+2}$ ;




$\text{Area} = \iint_S 1\, dS = \iint_R \sec\gamma\, dA$

$= \int_0^2 \int_0^{3x} \sqrt{x^2+2}\, dy\, dx = \int_0^2 \left( \sqrt{x^2+2} \cdot y \Big|_{y=0}^{y=3x} \right) dx$

$= \int_0^2 3x\sqrt{x^2+2}\,dx = \cancel{3}\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}(x^2+2)^{3/2}\Big|_0^2$

$= 6^{3/2} - 2^{3/2}$

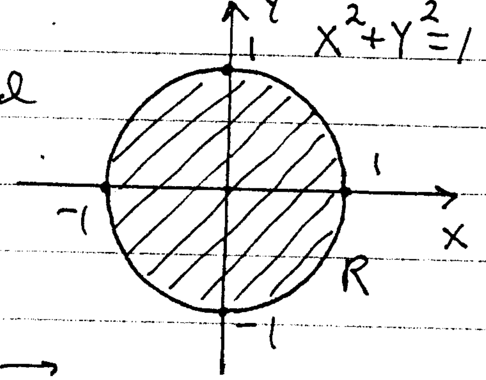
42.) $S$ : $x^2+y^2+z^2 = 2$ cut by $z = \sqrt{x^2+y^2}$ $\rightarrow$

$x^2+y^2 = 2 - z^2$ so $z = \sqrt{2-z^2}$ $\rightarrow$

$z^2 = 2 - z^2 \rightarrow 2z^2 = 2 \rightarrow z = 1 \rightarrow$

intersection of curves is

$x^2+y^2 = 1$ :




$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$ and

$\sec\gamma = \dfrac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{|2z|}$

$= \dfrac{\sqrt{4(x^2+y^2+z^2)}}{2z} = \dfrac{\sqrt{4(2)}}{2z} \rightarrow$

$\sec\gamma = \dfrac{\sqrt{2}}{z}$ ; then $z = \sqrt{2-(x^2+y^2)}$ and

$\text{Area} = \iint_S 1\,dS = \iint_R \sec\gamma\,dA$

$= \int_0^{2\pi}\int_0^1 \dfrac{\sqrt{2}}{\sqrt{2-r^2}}\cdot r\,dr\,d\theta$

$= \int_0^{2\pi} \sqrt{2}\cdot(-1)(2-r^2)^{1/2}\Big|_{r=0}^{r=1}\,d\theta$

$= \int_0^{2\pi}\left[-\sqrt{2}\,(1)^{1/2} - \sqrt{2}\,(2)^{1/2}\right]d\theta$

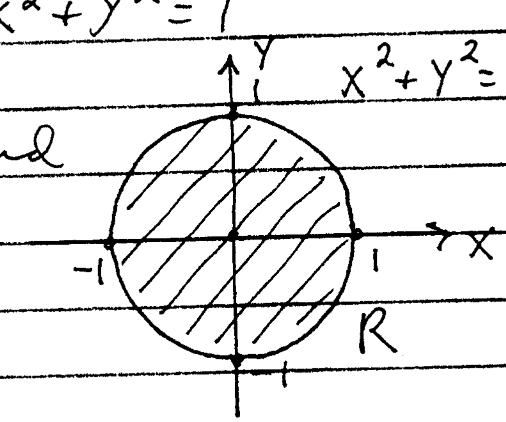
$= \int_0^{2\pi}(2-\sqrt{2})\,d\theta = (2-\sqrt{2})\,\theta\Big|_0^{2\pi} = (4-2\sqrt{2})\pi.$

43) $S$: $z = 3x$ cut by $x^2 + y^2 = 1$

$\rightarrow 3x - z = 0$ so

$\vec{\nabla} f = (3)\,\vec{i} + (0)\,\vec{j} + (-1)\,\vec{k}$ and

$$\sec\gamma = \frac{\sqrt{(3)^2 + (-1)^2}}{|-1|} = \sqrt{10},$$




then

$$\text{Area} = \iint_S 1\, dS = \iint_R \sec\gamma \cdot dA$$

$$= \sqrt{10} \cdot \iint_R 1\, dA = \sqrt{10} \cdot \pi (1)^2 = \sqrt{10}\,\pi$$

48) $S$: $2x^{3/2} + 2y^{3/2} - 3z = 0$

$R$: square $0 \le x \le 1,\ 0 \le y \le 1$, then

$\vec{\nabla} f = (3\sqrt{x})\,\vec{i} + (3\sqrt{y})\,\vec{j} + (-3)\,\vec{k}$, so that

$$\sec\gamma = \frac{\sqrt{(3\sqrt{x})^2 + (3\sqrt{y})^2 + (-3)^2}}{|-3|}$$

$$= \frac{1}{3}\sqrt{9(x+y+1)} = \sqrt{x+y+1} \quad ; \text{ then}$$

$$\text{Area} = \iint_S 1\, dS = \iint_R \sec\gamma \cdot dA$$

$$= \int_0^1 \int_0^1 \sqrt{x+y+1}\; dy\, dx$$

$$= \int_0^1 \frac{2}{3}(x+y+1)^{3/2}\Big|_{y=0}^{y=1} dx$$

$$= \int_0^1 \left[ \frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} \right] dx$$

$$= \left( \frac{2}{3} \cdot \frac{2}{5} (x+2)^{5/2} - \frac{2}{3} \cdot \frac{2}{5} (x+1)^{5/2} \right) \Big|_0^1$$

$$= \frac{4}{15} (3)^{5/2} - \frac{4}{15} (1)^{5/2} = \frac{4}{15} \left( 3^{5/2} - 1 \right).$$