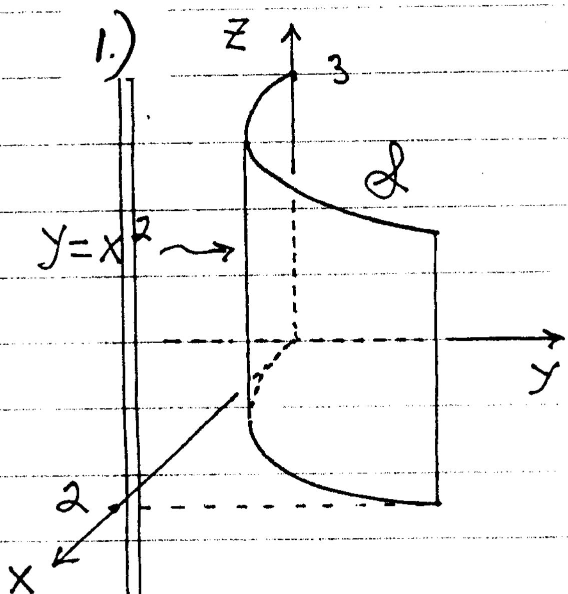
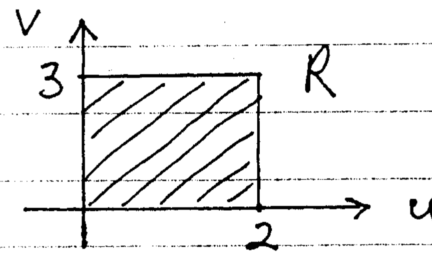
Section 16.6

1.) 

$$S: \begin{cases} x = u \\ y = u^2 \\ z = v \end{cases}$$

for $0 \le u \le 2$, $0 \le v \le 3$



$\vec{r}_u = (1)\,\vec{i} + (2u)\,\vec{j} + (0)\,\vec{k}$;

$\vec{r}_v = (0)\,\vec{i} + (0)\,\vec{j} + (1)\,\vec{k}$;

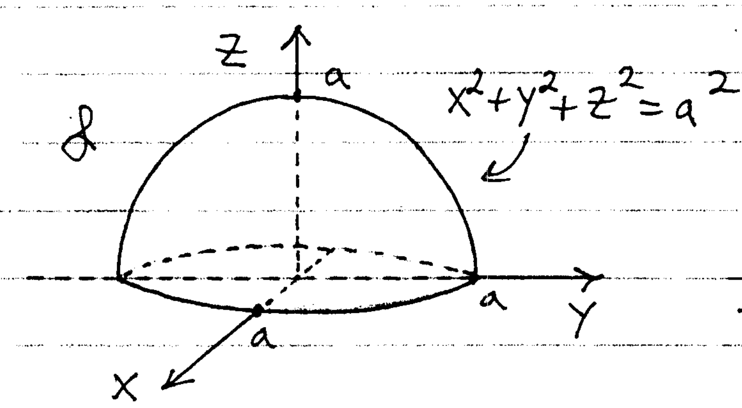
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

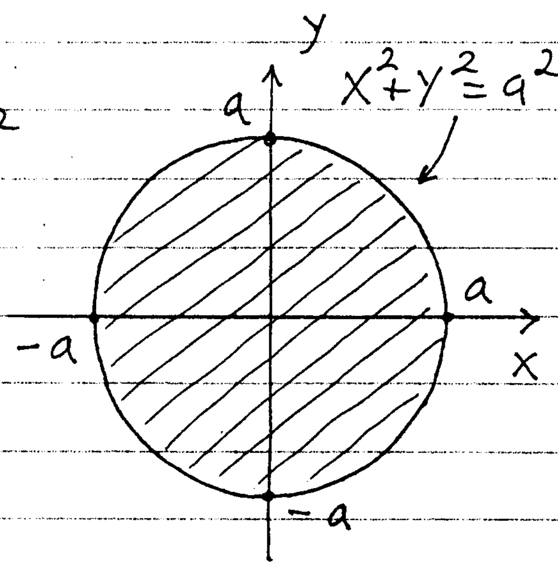
$$= (2u-0)\vec{i} - (1-0)\vec{j} + (0-0)\vec{k}$$
$$= (2u)\vec{i} + (-1)\vec{j} \; ;$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(2u)^2+(-1)^2} = \sqrt{4u^2+1} \; ; \text{ then}$$

$$\iint_{\mathcal{S}} x \, dS = \iint_R x \cdot |\vec{r}_u \times \vec{r}_v| \, dA$$
$$= \int_0^2 \int_0^3 u \cdot \sqrt{4u^2+1} \, dv \, du$$
$$= \int_0^2 \left(u\sqrt{4u^2+1} \cdot v \Big|_{v=0}^{v=3} \right) du$$
$$= \int_0^2 3u\sqrt{4u^2+1} \, du$$
$$= 3\frac{1}{8}\cdot\frac{2}{3}(4u^2+1)^{3/2}\Big|_0^2 = 3\frac{1}{12}(17)^{3/2} - 3\frac{1}{12}(1)^{3/2}$$
$$= \frac{1}{4}(17)^{3/2} - \frac{1}{4}$$

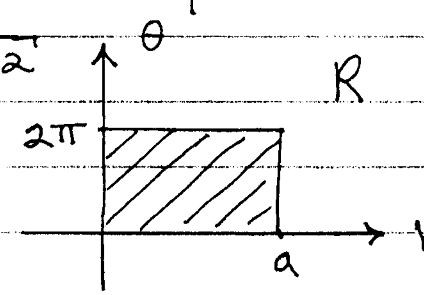
4.)





$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{a^2-(x^2+y^2)} = \sqrt{a^2-r^2} \end{cases}$$

for $0 \le r \le a$,
$0 \le \theta \le 2\pi$;



$$\vec{r}_\theta = (-r\sin\theta)\,\vec{i} + (r\cos\theta)\,\vec{j} + (0)\,\vec{k},$$

$$\vec{r}_r = (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + \tfrac{1}{2}(a^2-r^2)^{-\frac{1}{2}}\cdot -2r\cdot\vec{k}$$

$$= (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + \frac{-r}{\sqrt{a^2-r^2}}\cdot\vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{a^2-r^2}} \end{vmatrix}$$

$$= \left(\frac{-r^2\cos\theta}{\sqrt{a^2-r^2}}\right)\vec{i} - \left(\frac{-r^2\sin\theta}{\sqrt{a^2-r^2}}\right)\vec{j}$$

$$+ (-r\sin^2\theta - r\cos^2\theta)\,\vec{k}$$

$$= \left(\frac{-r^2\cos\theta}{\sqrt{a^2-r^2}}\right)\vec{i} + \left(\frac{r^2\sin\theta}{\sqrt{a^2-r^2}}\right)\vec{j} - r(\underbrace{\sin^2\theta+\cos^2\theta}_{1})\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{\frac{r^4\cdot\cos^2\theta}{a^2-r^2} + \frac{r^4\sin^2\theta}{a^2-r^2} + r^2}$$

$$= \sqrt{\frac{r^4}{a^2-r^2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) + r^2}$$

$$= \sqrt{\frac{r^4}{a^2-r^2} + r^2\cdot\frac{a^2-r^2}{a^2-r^2}}$$

$$= \sqrt{\frac{\cancel{r^4}+a^2r^2-\cancel{r^4}}{a^2-r^2}} = \frac{ar}{\sqrt{a^2-r^2}};$$ ~~then~~

$$\iint_S z^2\,dS = \iint_R z^2\cdot|\vec{r}_\theta \times \vec{r}_r|\,dA$$

$$= \int_0^{2\pi}\int_0^{a} (a^2 - r^2)\cdot \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta$$

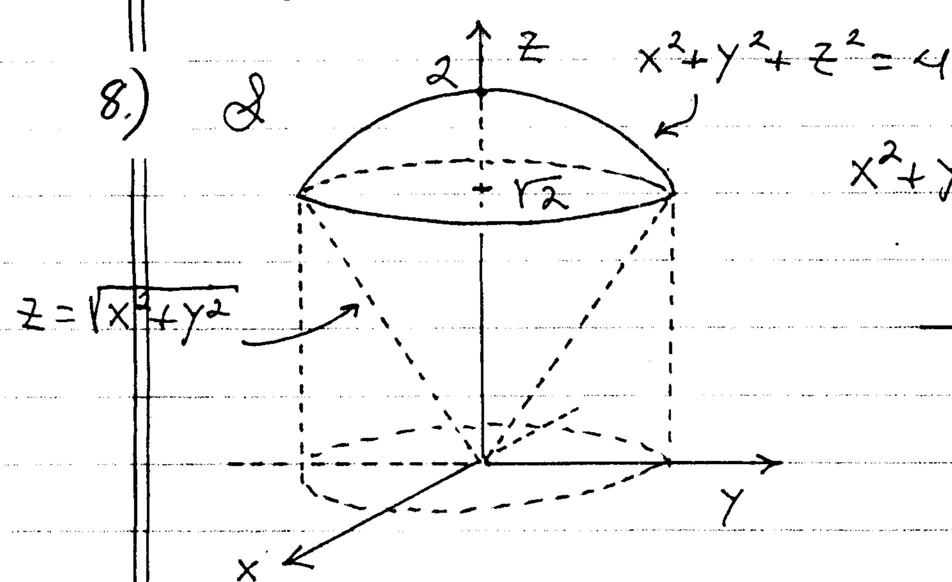
$$= \int_0^{2\pi}\int_0^{a} ar\sqrt{a^2-r^2}\, dr\, d\theta$$

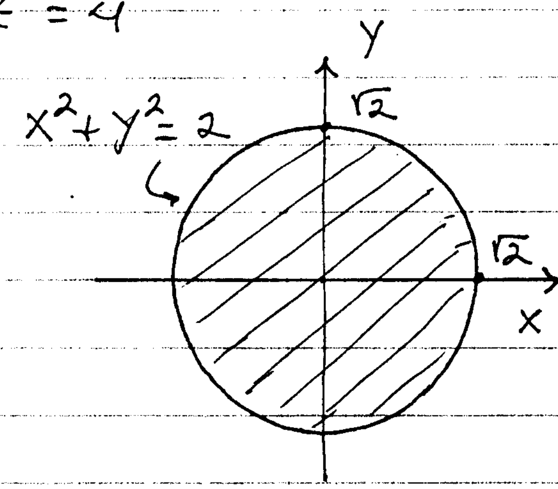
$$= \int_0^{2\pi} a\cdot \frac{-1}{\cancel{2}}\cdot \frac{\cancel{2}}{3}(a^2-r^2)^{3/2}\Big|_{r=0}^{r=a}\, d\theta$$

$$= \int_0^{2\pi} \left[a\cdot \frac{-1}{3}(0)^{3/2} - a\frac{-1}{3}(a^2)^{3/2}\right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}a^4\, d\theta = \frac{1}{3}a^4\cdot \theta\Big|_0^{2\pi} = \frac{2}{3}a^4\pi$$

8) $\mathcal{S}$



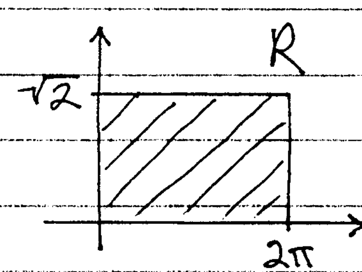


$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 4 \longrightarrow$$

$$x^2+y^2+x^2+y^2 = 4 \longrightarrow 2(x^2+y^2) = 4 \longrightarrow$$

$x^2+y^2 = 2$ and $z = \sqrt{2}$;

$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{4-(x^2+y^2)} = \sqrt{4-r^2} \end{cases}$$



for $0 \le r \le \sqrt{2}$,
$0 \le \theta \le 2\pi$

(SEE problem 30 solution.)

$$|\vec{r}_\theta \times \vec{r}_r| = \frac{2r}{\sqrt{4-r^2}} \text{ ; then}$$

$$\iint_{\mathcal{S}} yz\, dS = \iint_R yz\, |\vec{r}_\theta \times \vec{r}_r|\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (r\sin\theta)\cdot\cancel{\sqrt{4-r^2}}\cdot\frac{2r}{\cancel{\sqrt{4-r^2}}}\, dr\, d\theta$$

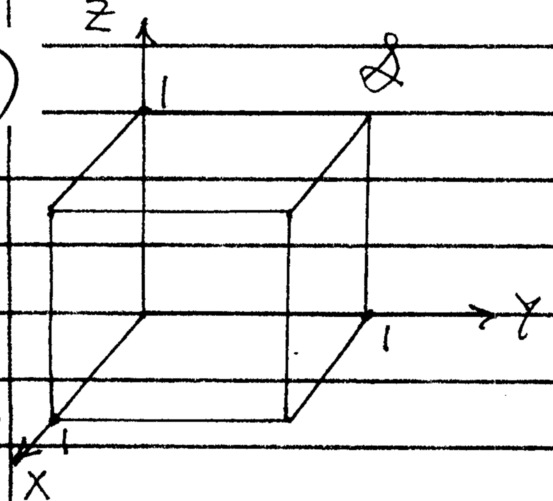
$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 2r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{2}{3}r^3\sin\theta\Big|_{r=0}^{r=\sqrt{2}}\right) d\theta$$

$$= \int_0^{2\pi}\frac{2}{3}\,2\sqrt{2}\,\sin\theta\, d\theta = -\frac{4}{3}\sqrt{2}\cos\theta\Big|_0^{2\pi}$$

$$= -\frac{4}{3}\sqrt{2}\left(\overset{1}{\cancel{\cos 2\pi}} - \overset{1}{\cancel{\cos 0}}\right) = 0$$

9.)



$$\iint_{\mathcal{S}} g(P)\, dS$$

$$= \iint_{\text{bottom}} (x+y+z)\, dS$$

$$= \iint_{\text{top}} (x+y+z)\, dS$$

$$= \iint_{\text{left}} (x+y+z)\, dS + \iint_{\text{right}} (x+y+z)\, dS$$

$$+ \iint_{\text{front}} (x+y+z)\, dS + \iint_{\text{back}} (x+y+z)\, dS$$

$$= \int_0^1\int_0^1 (x+y+0)\, dy\, dx + \int_0^1\int_0^1 (x+y+1)\, dy\, dx$$

$$+ \int_0^1\int_0^1 (x+0+z)\, dz\, dx + \int_0^1\int_0^1 (x+1+z)\, dz\, dx$$

$$+ \int_0^1\int_0^1 (1+y+z)\, dz\, dy + \int_0^1\int_0^1 (0+y+z)\, dz\, dy$$

$$= 3\int_0^1\int_0^1 (x+y)\, dy\, dx + 3\int_0^1\int_0^1 (x+y+1)\, dy\, dx$$

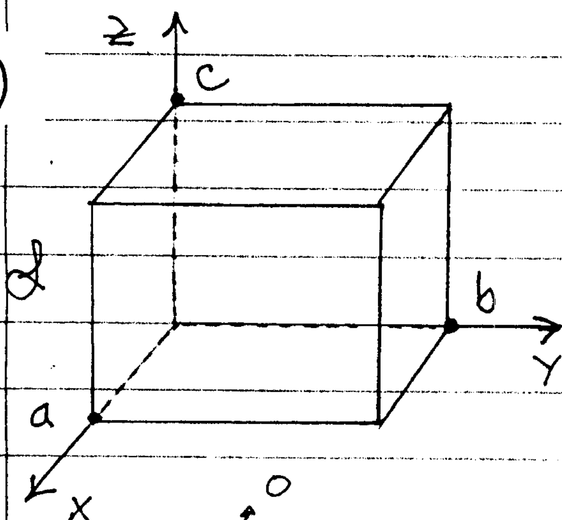
$$= 3\int_0^1 \left(xy + \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1} dx + 3\int_0^1 \left(xy + \tfrac{1}{2}y^2 + y\right)\Big|_{y=0}^{y=1} dx$$

$$= 3\int_0^1 \left(x+\tfrac{1}{2}\right) dx + 3\int_0^1 \left(x+\tfrac{1}{2}+1\right) dx$$

$$= 3\left(\tfrac{1}{2}x^2+\tfrac{1}{2}x\right)\Big|_0^1 + 3\left(\tfrac{1}{2}x^2+\tfrac{3}{2}x\right)\Big|_0^1$$

$$= 3\left(\tfrac{1}{2}+\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}+\tfrac{3}{2}\right) = 3(1)+3(2) = 9$$

11.)



$$\iint_S g(P)\,dS$$

$$= \iint_{\text{bottom}} xy\cancel{z}^{\,0}\,dS + \iint_{\text{top}} xyz\,dS$$

$$+ \iint_{\text{left}} x\cancel{y}^{\,0}z\,dS + \iint_{\text{right}} xyz\,dS$$

$$+ \iint_{\text{front}} xyz\,dS + \iint_{\text{back}} \cancel{x}^{\,0}yz\,dS$$

$$= \int_0^a\int_0^b xy(c)\,dy\,dx + \int_0^a\int_0^c x(b)z\,dz\,dx$$

$$+ \int_0^b\int_0^c (a)yz\,dz\,dy$$

$$= \int_0^a \left(cx\cdot\tfrac{1}{2}y^2\Big|_{y=0}^{y=b}\right) dx$$

$$+ \int_0^a \left(bx\cdot\tfrac{1}{2}z^2\Big|_{z=0}^{z=c}\right) dx$$

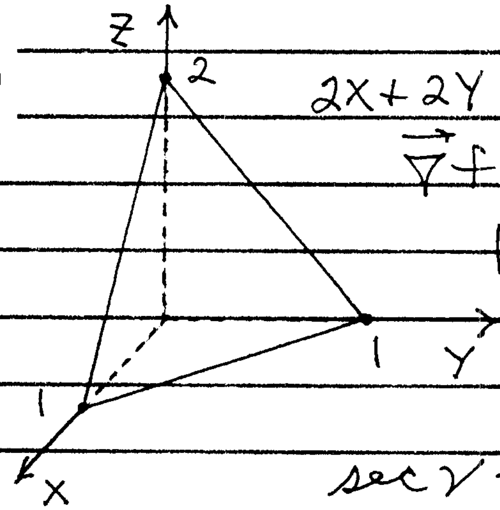
$$+ \int_0^b \left(ay\cdot\tfrac{1}{2}z^2\Big|_{z=0}^{z=c}\right) dy$$

$$= \int_0^a \frac{1}{2} b^2 c x \, dx + \int_0^a \frac{1}{2} b c^2 x \, dx + \int_0^b \frac{1}{2} a c^2 y \, dy$$

$$= \frac{1}{2} b^2 c \cdot \frac{1}{2} x^2 \Big|_0^a + \frac{1}{2} b c^2 \cdot \frac{1}{2} x^2 \Big|_0^a + \frac{1}{2} a c^2 \cdot \frac{1}{2} y^2 \Big|_0^b$$

$$= \frac{1}{4} a^2 b^2 c + \frac{1}{4} a^2 b c^2 + \frac{1}{4} a b^2 c^2$$

13)



$2x + 2y + z = 2$,

$\vec{\nabla} f = (2)\vec{i} + (2)\vec{j} + (1)\vec{k}$ and

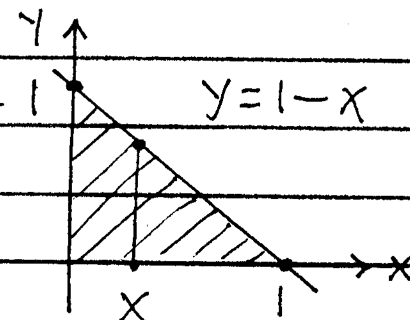
$|\vec{\nabla} f| = \sqrt{(2)^2 + (2)^2 + (1)^2}$

$= \sqrt{9} = 3$,

$\sec \gamma = \dfrac{|\vec{\nabla} f|}{|f_z|} = \dfrac{3}{1} = 3$; then

$$\iint_S g(P)\, dS = \iint_R (x + y + z) \cdot \sec\gamma \cdot dA$$

$$= \int_0^1 \int_0^{1-x} (x + y + (2 - 2x - 2y)) \cdot 3 \, dy \, dx$$



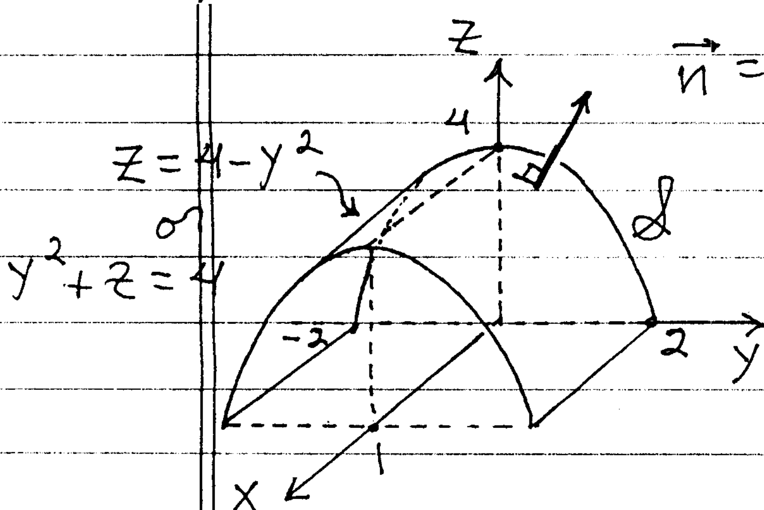
$$= 3 \int_0^1 \int_0^{1-x} (2 - x - y) \, dy \, dx$$

$$= 3 \int_0^1 \left(2y - xy - \frac{1}{2} y^2\right) \Big|_{y=0}^{y=1-x} dx$$

$$= 3 \int_0^1 \left(2(1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right) dx$$

$$= 3\int_0^1 \left(2-2x-x+x^2-\tfrac{1}{2}(x^2-2x+1)\right)dx$$
$$= 3\int_0^1 \left(2-3x+x^2-\tfrac{1}{2}x^2+x-\tfrac{1}{2}\right)dx$$
$$= 3\int_0^1 \left(\tfrac{1}{2}x^2-2x+3/2\right)dx$$
$$= 3\left(\tfrac{1}{6}x^3-x^2+\tfrac{3}{2}x\right)\Big|_0^1$$
$$= 3\left(\tfrac{1}{6}-1+3/2\right) = 3\left(\tfrac{1}{6}-\tfrac{6}{6}+\tfrac{9}{6}\right)$$
$$= 3\left(\tfrac{4}{6}\right) = 2$$

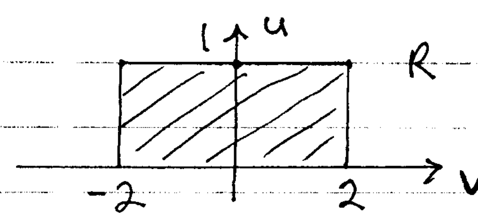
19.) $\vec{\nabla} f = (0)\vec{i} + (2y)\vec{j} + (1)\vec{k}$ and so



$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2y)\vec{j} + (1)\vec{k}}{\sqrt{(2y)^2+(1)^2}}$$
$$= \frac{2y}{\sqrt{4y^2+1}}\vec{j} + \frac{1}{\sqrt{4y^2+1}}\vec{k},$$

$$\vec{F}(x,y,z) = (z^2)\vec{i} + (x)\vec{j} + (-3z)\vec{k};$$

$$S: \begin{cases} x = u \\ y = v \\ z = 4 - v^2 \end{cases} \quad \text{for } 0 \le u \le 1, \; -2 \le v \le 2$$



$\vec{r}_u = (1)\vec{i} + (0)\vec{j} + (0)\vec{k}$, $\vec{r}_v = (0)\vec{i} + (1)\vec{j} + (-2v)\vec{k}$,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2v \end{vmatrix}$$

$= (0)\vec{i} - (-2v - 0)\vec{j} + (1-0)\vec{k} = (2v)\vec{j} + (1)\vec{k}$;

~~then~~ $|\vec{r}_u \times \vec{r}_v| = \sqrt{(2v)^2 + (1)^2} = \sqrt{4v^2+1}$; then

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \left(\frac{2xy}{\sqrt{4y^2+1}} + \frac{-3z}{\sqrt{4y^2+1}} \right) dS$$

$$= \iint_R \frac{2xy - 3z}{\sqrt{4y^2+1}} \cdot |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \int_0^1 \int_{-2}^{2} \frac{2uv - 3(4-v^2)}{\cancel{\sqrt{4v^2+1}}} \cdot \cancel{\sqrt{4v^2+1}}\, dv\, du$$

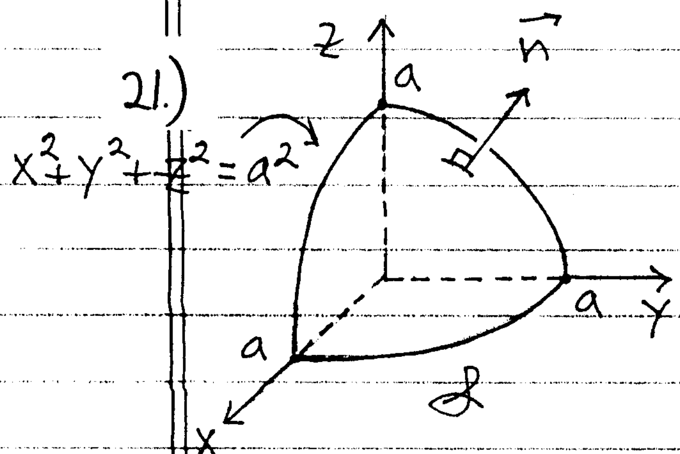
$$= \int_0^1 \int_{-2}^{2} (2uv - 12 + 3v^2)\, dv\, du$$

$$= \int_0^1 (uv^2 - 12v + v^3)\Big|_{v=-2}^{v=2}\, du$$

$$= \int_0^1 \left((\cancel{4u} - 24 + 8) - (\cancel{4u} + 24 - 8) \right) du$$

$$= \int_0^1 -32\, du = -32u \Big|_0^1 = -32$$

21.)



$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$,

$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$

$= \sqrt{4(x^2+y^2+z^2)}$

$= 2\sqrt{a^2}$

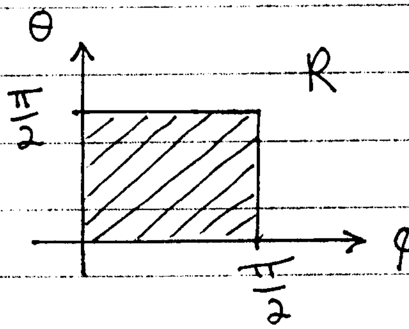
$= 2a$, so

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k} \; ;$$

(Use spherical coordinates.)

$$\mathcal{S}: \begin{cases} x = a \sin\phi \cos\theta \\ y = a \sin\phi \sin\theta \\ z = a \cos\phi \end{cases}$$

for $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$



$\vec{r}_\theta = (-a \sin\phi \sin\theta)\,\vec{i} + (a \sin\phi \cos\theta)\,\vec{j}$,

$\vec{r}_\phi = (a \cos\phi \cos\theta)\,\vec{i} + (a \sin\phi \cos\theta)\,\vec{j} + (-a \sin\phi)\,\vec{k}$, and

(See Parametrized Surfaces handout.)

... $|\vec{r}_\theta \times \vec{r}_\phi| = a^2 \sin\phi$; then $\vec{F} = z\vec{k}$ and

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{S}} \frac{z^2}{a}\, dS$$

$$= \iint_R \frac{z^2}{a} \cdot |\vec{r}_\theta \times \vec{r}_\phi|\, dA = \int_0^{\pi/2}\!\!\int_0^{\frac{\pi}{2}} \frac{a^2 \cos^2\phi}{a} \cdot a^2 \sin\phi\, d\phi\, d\theta$$

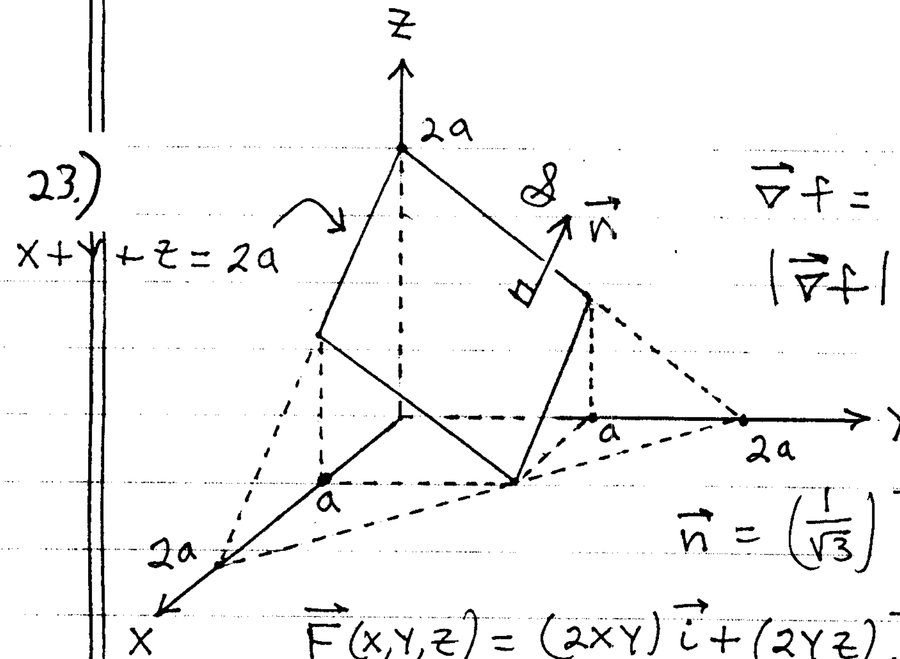
$$= \int_0^{\frac{\pi}{2}}\!\!\int_0^{\frac{\pi}{2}} a^3 \cos^2\phi \sin\phi \;\; d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3} a^3 \cos^3\phi\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\left(\frac{-1}{3} a^3 \overset{0}{\cancel{\cos}}{}^3 \frac{\pi}{2}\right) - \left(\frac{-1}{3} a^3 \overset{1}{\cancel{\cos}}{}^3 0\right)\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} a^3\, d\theta = \frac{1}{3} a^3 \cdot \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{6} a^3 \pi$$

23.) $x+y+z=2a$



$\vec{\nabla} f = (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$,

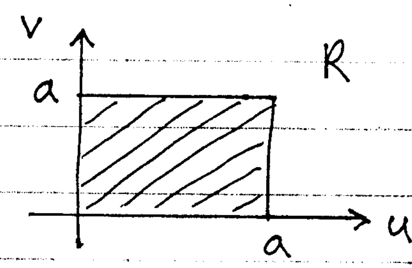
$|\vec{\nabla} f| = \sqrt{3}$;

$\vec{n} = \left(\frac{1}{\sqrt{3}}\right)\vec{i} + \left(\frac{1}{\sqrt{3}}\right)\vec{j} + \left(\frac{1}{\sqrt{3}}\right)\vec{k}$,

$\vec{F}(x,y,z) = (2xy)\vec{i} + (2yz)\vec{j} + (2xz)\vec{k}$;

$$\mathcal{S}: \begin{cases} x = u \\ y = v \\ z = 2a - x - y = 2a - u - v \end{cases}$$

for $0 \le u \le a$, $0 \le v \le a$;



$\vec{r}_u = (1)\vec{i} + (0)\vec{j} + (-1)\vec{k}$,

$\vec{r}_v = (0)\vec{i} + (1)\vec{j} + (-1)\vec{k}$,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= (0-{-1})\vec{i} - (-1-0)\vec{j} + (1-0)\vec{k}$

$= (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$, $|\vec{r}_u \times \vec{r}_v| = \sqrt{3}$; then

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{S}} \frac{2}{\sqrt{3}}(xy + yz + xz)\, dS$$

$$= \iint_R \frac{2}{\sqrt{3}}\big(uv + v(2a-u-v) + u(2a-u-v)\big)\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \int_0^a \int_0^a \frac{2}{\cancel{\sqrt{3}}}\left(\cancel{uv} + 2av - \cancel{uv} - v^2 + 2au - u^2 - uv\right)\cancel{\sqrt{3}}\, dv\, du$$

$$= \int_0^a \int_0^a (4av - 2v^2 + 4au - 2u^2 - 2uv)\, dv\, du$$

$$= \int_0^a \left(2av^2 - \frac{2}{3}v^3 + 4auv - 2u^2 v - uv^2\right)\Big|_{v=0}^{v=a}$$

$$= \int_0^a \left[2a^3 - \frac{2}{3}a^3 + 4a^2 u - 2au^2 - a^2 u\right] du$$

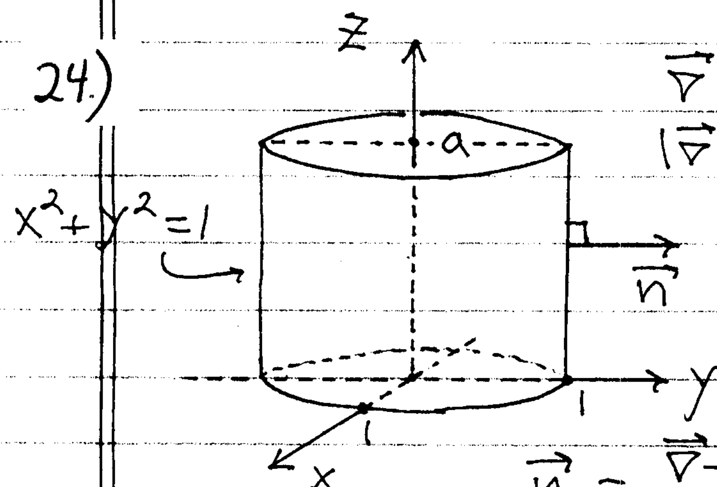
$$= \int_0^a \left[\frac{4}{3}a^3 + 3a^2 u - 2au^2\right] du$$

$$= \left(\frac{4}{3}a^3 u + 3a^2 \cdot \frac{1}{2}u^2 - 2a \cdot \frac{1}{3}u^3\right)\Big|_0^a$$

$$= \frac{4}{3}a^4 + \frac{3}{2}a^4 - \frac{2}{3}a^4$$

$$= \frac{8}{6}a^4 + \frac{9}{6}a^4 - \frac{4}{6}a^4 = \frac{13}{6}a^4$$

24)



$$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j},$$

$$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2}$$

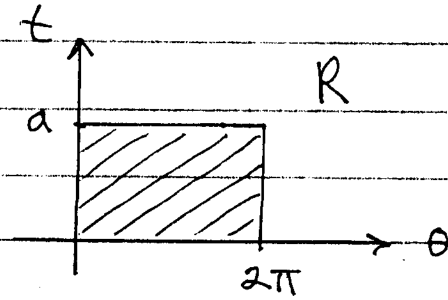
$$= \sqrt{4x^2 + 4y^2}$$

$$= \sqrt{4(x^2+y^2)}$$

$$= \sqrt{4(1)} = 2, \text{ so}$$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = (x)\vec{i} + (y)\vec{j}$$

$$S: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = t \end{cases}$$



for $0 \le \theta \le 2\pi$, $0 \le t \le a$;

$$\vec{r}_\theta = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j} + (0)\vec{k},$$

$$\vec{r}_t = (0)\vec{i} + (0)\vec{j} + (1)\vec{k} \;;$$

$$\vec r_\theta \times \vec r_t = \begin{vmatrix} \vec i & \vec j & \vec k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\cos\theta - 0)\vec i - (-\sin\theta - 0)\vec j + (0-0)\vec k$$

$$= (\cos\theta)\vec i + (\sin\theta)\vec j \;;$$

$$|\vec r_\theta \times \vec r_t| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1 \;;$$

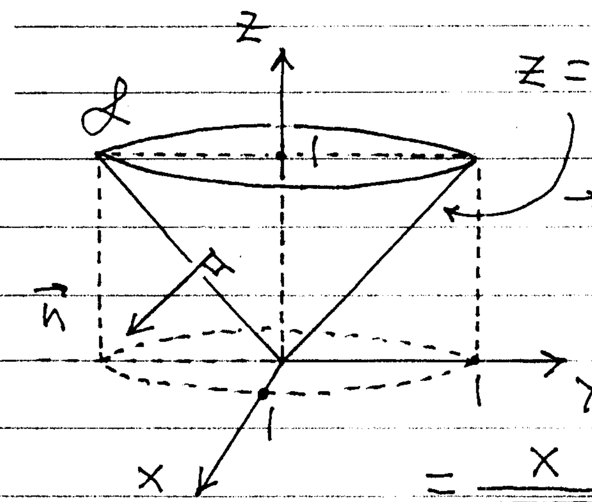
$$\cancel{\vec F(x,y,z) = (x)\vec i + (y)\vec j + (z)\vec k \;; \text{ then}}$$

$$\text{Flux} = \iint_{\mathcal S} \vec F \cdot \vec n \, dS = \iint_{\mathcal S} (x^2+y^2)\, dS$$

$$= \iint_R (1)\,|\vec r_\theta \times \vec r_t|\, dA = \int_0^{2\pi}\int_0^a 1 \, dt\, d\theta$$

$$= \text{area } R = 2a\pi \,.$$

25.)



$$z = \sqrt{x^2+y^2} \longrightarrow 0 = \sqrt{x^2+y^2} - z$$

$$\to \vec\nabla f = \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x \cdot \vec i + \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y \cdot \vec j + (-1)\vec k$$

$$= \frac{x}{\sqrt{x^2+y^2}}\vec i + \frac{y}{\sqrt{x^2+y^2}}\vec j + (-1)\vec k \;;$$

$$|\vec\nabla f| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + (-1)^2}$$

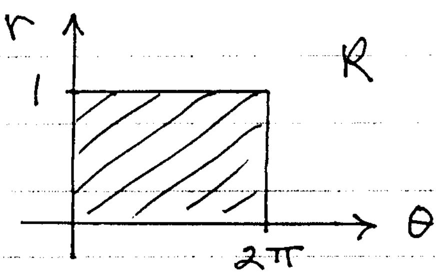
$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2}+1} = \sqrt{1+1} = \sqrt{2} \text{ ; so}$$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j} + (-1)\vec{k}\right);$$

$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{x^2+y^2} = \sqrt{r^2} = r \end{cases}$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi$;



$$\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + (0)\vec{k},$$
$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + (1)\vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$$= (r\cos\theta - 0)\vec{i} - (-r\sin\theta - 0)\vec{j} + (-r\sin^2\theta - r\cos^2\theta)\vec{k}$$

$$= (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + (-r)\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + (-r)^2}$$
$$= \sqrt{r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + r^2}$$
$$= \sqrt{2r^2} = \sqrt{2}\cdot r \text{ ; then}$$

$$\vec{F}(x,y,z) = (xy)\vec{i} + (-z)\vec{k} \quad \text{and}$$

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, dS = \iint_{\mathcal{S}} \frac{1}{\sqrt{2}}\left[\frac{x^2y}{\sqrt{x^2+y^2}} + z\right] dS$$

$$= \iint \frac{1}{\sqrt{2}} \left[\frac{x^2 y}{\sqrt{x^2+y^2}} + z \right] \cdot \left|\vec{r}_\theta \times \vec{r}_r\right| \, dA$$

$$\overset{R}{=} \int_0^{2\pi}\int_0^1 \cancel{\frac{1}{\sqrt{2}}} \left[\frac{(r^2\cos^2\theta)\cdot(r\sin\theta)}{\sqrt{r^2}} + r \right] \cancel{\sqrt{2}}\, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{\overset{2}{\cancel{r^3}}\cos^2\theta\sin\theta}{\cancel{r}} + r \right] \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^3\cos^2\theta\sin\theta + r^2 \right) dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}r^4\cos^2\theta\sin\theta + \frac{1}{3}r^3 \right) \Big|_{r=0}^{r=1} d\theta$$

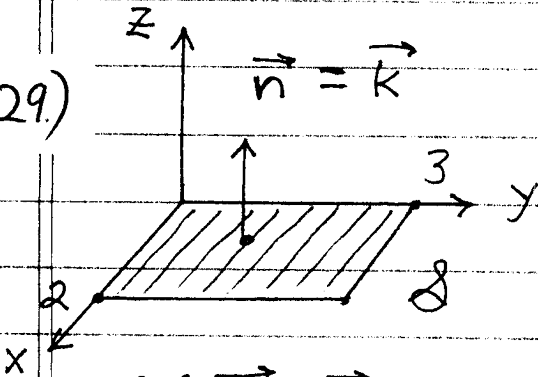
$$= \int_0^{2\pi} \left(\frac{1}{4}\cos^2\theta\sin\theta + \frac{1}{3} \right) d\theta$$

$$= \left(\frac{1}{4}\cdot\frac{-1}{3}\cos^3\theta + \frac{1}{3}\theta \right) \Big|_0^{2\pi}$$

$$= \left(\frac{-1}{12}\overset{1}{\cancel{\cos}}{}^3 2\pi + \frac{2}{3}\pi \right) - \left(\frac{-1}{12}\overset{1}{\cancel{\cos}}{}^3 0 - 0 \right)$$

$$= \frac{2}{3}\pi$$

29)



$\vec{F}(x,y,z) = (-1)\vec{i} + (2)\vec{j} + (3)\vec{k}$,

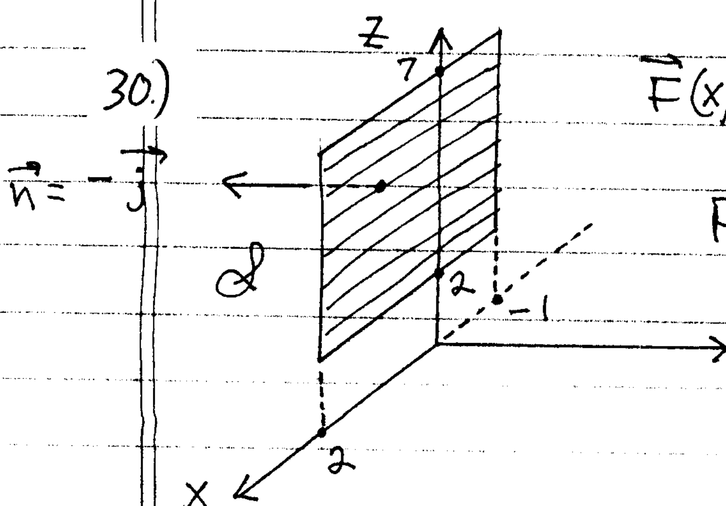
$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n} \, dS$$

$$= \iint_{\mathcal{S}} \vec{F} \cdot \vec{k} \, dS = \int_0^2 \int_0^3 3 \, dy \, dx$$

$$= 3 \left(\int_0^2 \int_0^3 1 \, dy \, dx \right) = 3 \, (\text{area } \mathcal{S})$$

$$= 3(6) = 18$$

30)



$\vec{F}(x,y,z) = (x^2 y)\vec{i} + (-2)\vec{j} + (xz)\vec{k}$,

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n} \, dS$$

$$= \iint_{\mathcal{S}} \vec{F} \cdot (-\vec{j}) \, dS$$

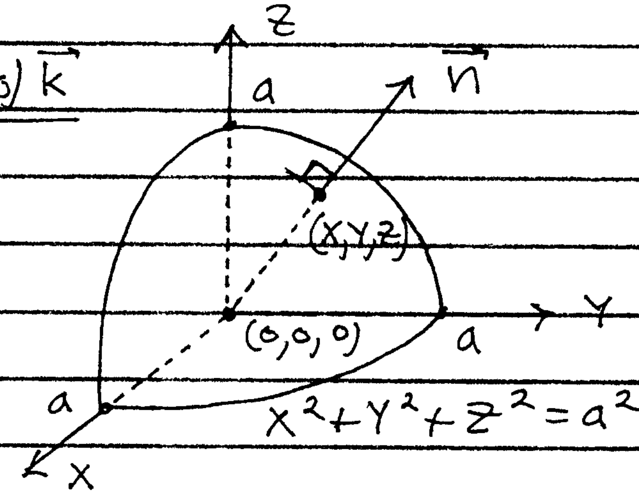
$= \int_{-1}^{2}\int_{2}^{7} 2 \, dz\, dx = 2\left(\int_{-1}^{2}\int_{2}^{7} 1 \, dz\, dx\right)$

$= 2\,(\text{area } \mathcal{S}) = 2(15) = 30$.

31.) $\vec{n} = \dfrac{(x-0)\vec{i} + (y-0)\vec{j} + (z-0)\vec{k}}{\sqrt{x^2+y^2+z^2}}$

$= \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{a^2}}$

$= \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k}$;



$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$ and

$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$

$= \sqrt{4(x^2+y^2+z^2)} = \sqrt{4(a^2)} = 2a$, so

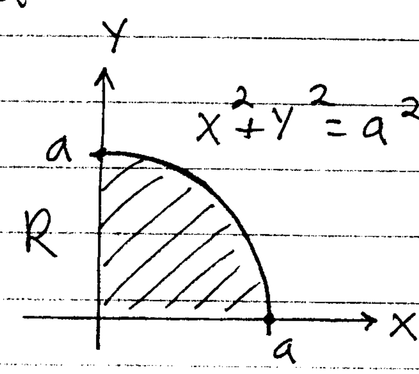
$\sec \gamma = \dfrac{|\vec{\nabla} f|}{|f_z|} = \dfrac{2a}{2z} = \dfrac{a}{z}$; then

$\vec{F}(x,y,z) = z\,\vec{k}$ and

$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n} \, dS = \iint_{R} \frac{z^2}{a} \sec\gamma \, dA$

$= \iint_{R} \frac{z^2}{a} \cdot \frac{a}{z} \, dA$

$= \iint_{R} z \, dA$



$$= \iint_R \sqrt{a^2 - (x^2+y^2)}\ dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{a} \sqrt{a^2 - r^2} \cdot r\ dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2}{3} \cdot \frac{-1}{2} (a^2 - r^2)^{3/2} \Big|_{r=0}^{r=a}\ d\theta$$

$$= \int_0^{\pi/2} \left(-\frac{1}{3}(0)^{3/2} - \frac{-1}{3}(a^2)^{3/2} \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} a^3\, d\theta = \frac{1}{3} a^3 \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{6} a^3 \pi$$

34.) $\vec{F}(x,y,z) = (xz)\vec{i} + (yz)\vec{j} + (z^2)\vec{k}$,

$\vec{n} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k}$, $\sec \gamma = a/z$;

then Flux $= \iint_S \vec{F} \cdot \vec{n}\ dS$

$$= \iint_R \left(\frac{x^2 z}{a} + \frac{y^2 z}{a} + \frac{z^3}{a} \right) \cdot \sec \gamma\ dA$$

$$= \iint_R \frac{\cancel{z}}{\cancel{a}} \underbrace{(x^2 + y^2 + z^2)}_{a^2} \cdot \frac{\cancel{a}}{\cancel{z}}\ dA$$

$$= a^2 \iint_R 1\ dA = a^2\ (\text{area } R) = a^2 \cdot \frac{1}{4} \pi a^2$$

$$= \frac{1}{4} a^4 \pi$$

35.) $\vec{F}(x,y,z) = (x)\vec{i} + (y)\vec{j} + (z)\vec{k}$,

$\vec{n} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k}$, $\sec \gamma = a/z$;

then $\text{Flux} = \iint_{S} \vec{F} \cdot \vec{n} \, dS$

$$= \iint_{R} \left(\frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a}\right) \cdot \sec\gamma \, dA$$

$$= \iint_{R} \frac{1}{\cancel{a}}(x^2+y^2+z^2) \cdot \frac{\cancel{a}}{z} \, dA$$

$$= \iint_{R} \frac{a^2}{z} \, dA = \int_0^{\frac{\pi}{2}} \int_0^a \frac{a^2}{\sqrt{a^2-(x^2+y^2)}} \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a a^2 \cdot \frac{r}{\sqrt{a^2-r^2}} \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -a^2 \cdot (a^2-r^2)^{\frac{1}{2}} \Big|_{r=0}^{r=a} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-a^2 (0)^{\frac{1}{2}} - -a^2 (a^2)^{\frac{1}{2}}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^3 \, d\theta = a^3 \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2} a^3 \pi$$

36) $\vec{F}(x,y,z) = \dfrac{(x)\vec{i} + (y)\vec{j} + (z)\vec{k}}{\sqrt{x^2+y^2+z^2}}$,

$\vec{n} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k}$, $\sec\gamma = \frac{a}{z}$;

then
$\text{Flux} = \iint_{S} \vec{F} \cdot \vec{n} \, dS$

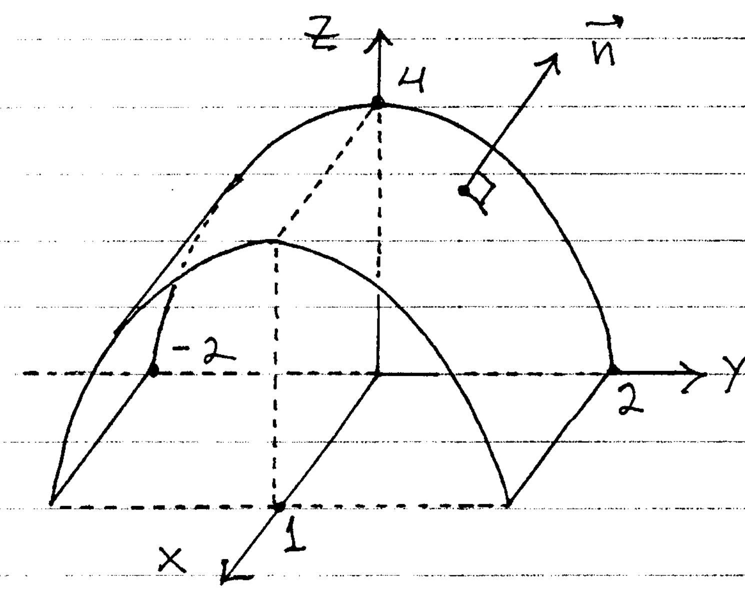
$$= \iint_{S} \frac{1}{a} \cdot \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} \, dS$$

$$= \iint_{S} \frac{1}{a} \cdot \frac{a^2}{\sqrt{a^2}} \, dS = \iint_{R} 1 \cdot \sec\gamma \, dA$$

$$= \iint_R \frac{a}{z}\, dA = \frac{1}{a} \iint_R \frac{a^2}{z}\, dA$$

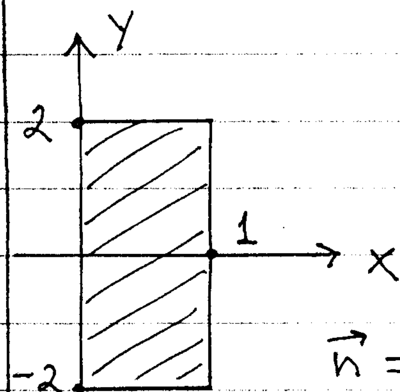
$$= \frac{1}{a}\left(\frac{1}{2} a^3 \pi\right) \qquad \text{(SEE solution 25.)}$$

$$= \frac{1}{2} a^2 \pi$$



37) $S: z = 4 - y^2$

or $y^2 + z = 4$,



$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2y)\vec{j} + (1)\vec{k}}{\sqrt{(2y)^2 + (1)^2}} \rightarrow$$

$$\vec{n} = \frac{2y}{\sqrt{4y^2+1}}\vec{j} + \frac{1}{\sqrt{4y^2+1}}\vec{k} \quad \text{and}$$

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|f_z|} = \frac{\sqrt{4y^2+1}}{1} = \sqrt{4y^2+1}\text{; then}$$

$$\vec{F}(x,y,z) = (z^2)\vec{i} + (x)\vec{j} + (-3z)\vec{k} \quad \text{and}$$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \left(\frac{2xy}{\sqrt{4y^2+1}} + \frac{-3z}{\sqrt{4y^2+1}}\right) dS$$

$$= \iint_R \frac{2xy - 3z}{\sqrt{4y^2+1}} \cdot \sec\gamma\, dA$$

$$= \int_0^1 \int_{-2}^{2} \frac{2xy - 3z}{\cancel{\sqrt{4y^2+1}}} \cdot \cancel{\sqrt{4y^2+1}}\, dy\, dx$$

$$= \int_0^1 \int_{-2}^{2} (2xy - 3(4-y^2))\, dy\, dx$$

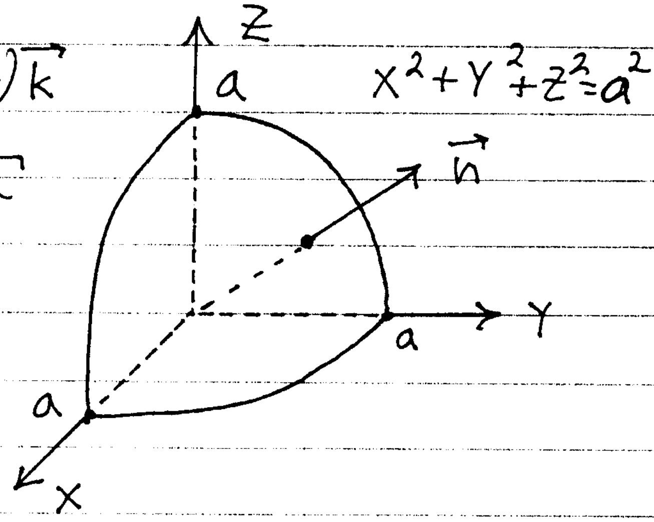
$$= \int_0^1 \int_{-2}^{2} (2xy - 12 + 3y^2)\, dy\, dx$$

$$= \int_0^1 (xy^2 - 12y + y^3) \Big|_{y=-2}^{y=2} dx$$

$$= \int_0^1 [(\cancel{4x} - 24 + 8) - (\cancel{4x} + 24 - 8)]\, dx$$

$$= \int_0^1 -32\, dx = -32x \Big|_0^1 = -32$$

43) $\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$



and

$$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$$

$$= \sqrt{4(x^2+y^2+z^2)}$$

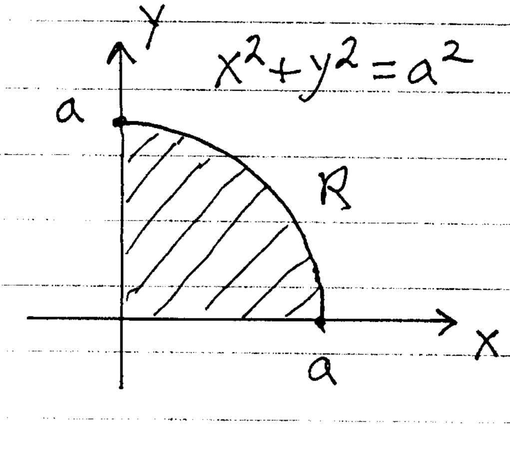
$$= \sqrt{4(a^2)} = 2a,$$

~~so~~

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k},$$

$$\sec \gamma = \frac{|\vec{\nabla} f|}{|f_z|} = \frac{2a}{2z} = \frac{a}{z}; \text{ then}$$

$\bar{x}$ for centroid is



$$\bar{x} = \frac{\iint_S x\, dS}{\iint_S 1\, dS};$$

$$\iint_S 1\, dS = \iint_R \sec \gamma\, dA$$

$$= \iint_R \frac{a}{z}\, dA = \iint_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-a \cdot (a^2-r^2)^{1/2} \Big|_{r=0}^{r=a} \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-a(0)^{1/2} - -a(a^2)^{1/2} \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^2\, d\theta = a^2 \theta \Big|_0^{\pi/2} = \frac{1}{2} a^2 \pi \;;$$

$$\iint_S x\, dS = \iint_R x \cdot \sec\gamma\, dA$$

$$= \iint_R x \cdot \frac{a}{z}\, dA = \iint_R \frac{ax}{\sqrt{a^2-(x^2+y^2)}}\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a \frac{ar\cos\theta}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta \quad \text{(TOO DIFFICULT!)}$$

By Symmetry we know that $\overline{x} = \overline{y} = \overline{z}$ and $\overline{z}$ is EASY to compute! Then

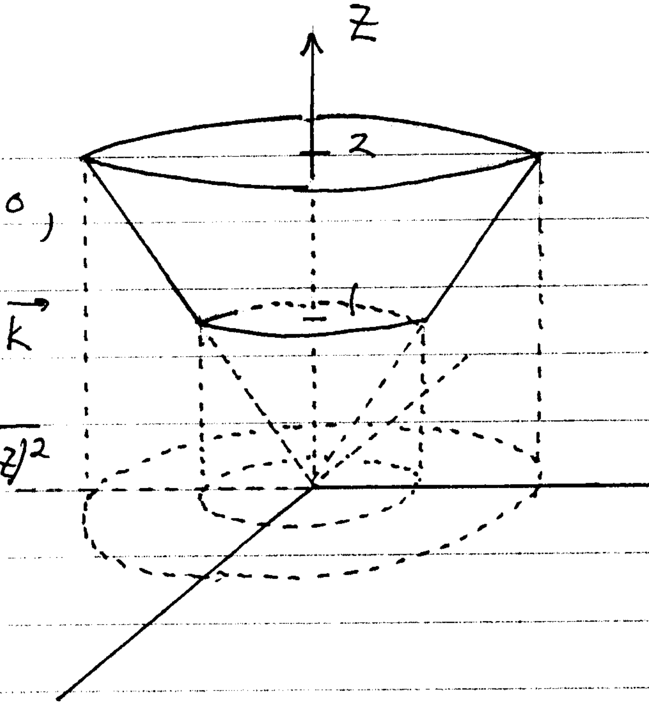
$$\iint_S z\, dS = \iint_R z \cdot \sec\gamma\, dA$$

$$= \iint_R \cancel{z} \cdot \frac{a}{\cancel{z}}\, dA = a \iint_R 1\, dA = a\,(\text{area } R)$$

$$= a \cdot \frac{1}{4}\pi a^2 = \frac{1}{4} a^3 \pi, \quad \text{so}$$

$$\overline{x} = \overline{z} = \frac{\frac{1}{4}a^3\pi}{\frac{1}{2}a^2\pi} = \frac{a}{2}\,.$$

45.) $\mathcal{S}: x^2+y^2-z^2=0$,



$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (-2z)\vec{k}$

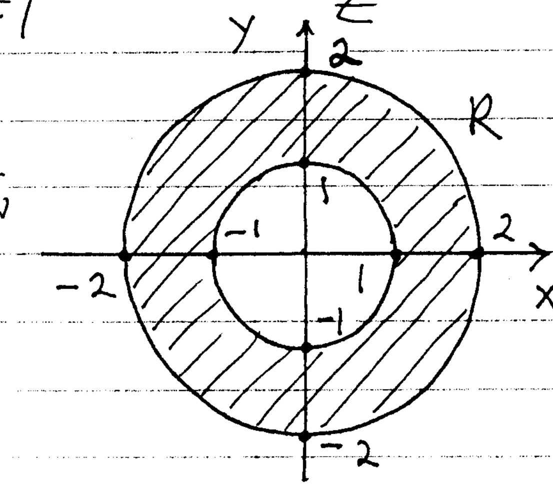
and

$|\vec{\nabla} f| = \sqrt{(2x)^2+(2y)^2+(-2z)^2}$

$= \sqrt{4(x^2+y^2+z^2)}$

$= 2\sqrt{x^2+y^2+z^2}$;

$$\sec\gamma = \frac{|\vec{\nabla} f|}{|f_z|} = \frac{2\sqrt{x^2+y^2+z^2}}{|-2z|} = \frac{\sqrt{x^2+y^2+z^2}}{z} ;$$



then

$$\bar{z} = \frac{\iint_{\mathcal{S}} z \cdot \cancel{\delta}\, dS}{\iint_{\mathcal{S}} \cancel{\delta}\, dS} ;$$

$$\iint_{\mathcal{S}} 1\, dS = \iint_R \sec\gamma\, dA$$

$$= \iint_R \frac{\sqrt{x^2+y^2+z^2}}{z}\, dA = \iint_R \frac{\sqrt{x^2+y^2+(x^2+y^2)}}{\sqrt{x^2+y^2}}\, dA$$

$$= \iint_R \frac{\sqrt{2(x^2+y^2)}}{\sqrt{x^2+y^2}}\, dA = \iint_R \sqrt{2} \cdot dA$$

$$= \sqrt{2} \iint_R 1\, dA = \sqrt{2}\,(\text{area } R)$$

$$= \sqrt{2}\left(\pi(2)^2 - \pi(1)^2\right) = 3\sqrt{2}\,\pi \text{ , and}$$

$$\iint_S z\,dS = \iint_R z\cdot \sec\gamma\,dS$$

$$= \iint_R \cancel{z}\cdot\frac{\sqrt{x^2+y^2+z^2}}{\cancel{z}}\,dA$$

$$= \iint_R \sqrt{x^2+y^2+(x^2+y^2)}\,dA$$

$$= \iint_R \sqrt{2}\cdot\sqrt{x^2+y^2}\,dA$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{2}\cdot\sqrt{r^2}\cdot r\,dr\,d\theta$$

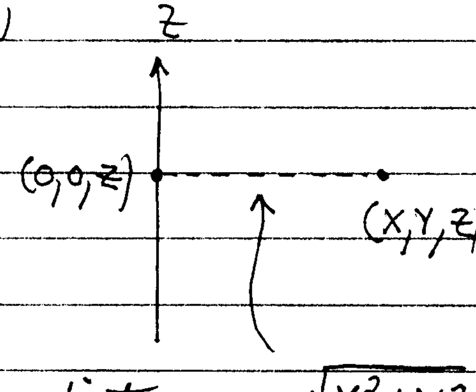
$$= \int_0^{2\pi}\int_1^2 \sqrt{2}\,r^2\,dr\,d\theta = \int_0^{2\pi}\left(\sqrt{2}\cdot\frac{1}{3}r^3\Big|_{r=1}^{r=2}\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{\sqrt{2}}{3}(8)-\frac{\sqrt{2}}{3}(1)\right)d\theta$$

$$= \int_0^{2\pi}\frac{7}{3}\sqrt{2}\,d\theta = \frac{7}{3}\sqrt{2}\,\theta\Big|_0^{2\pi}$$

$$= \frac{14}{3}\sqrt{2}\,\pi$$; then

$$\bar{z} = \frac{\frac{14}{3}\cancel{\sqrt{2}}\,\cancel{\pi}}{3\cancel{\sqrt{2}}\,\cancel{\pi}} = \frac{14}{9}$$;



M. of I. $= \iint_S (\text{distance})^2\,\delta\,dS$

$$= \delta\iint_S (x^2+y^2)\,dS$$

distance $= \sqrt{x^2+y^2}$

$$= \delta \iint_R (x^2+y^2) \cdot \sec\gamma \, dA$$

$$= \delta \iint_R (x^2+y^2) \cdot \frac{\sqrt{x^2+y^2+z^2}}{z} \, dA$$

$$= \delta \iint_R (x^2+y^2) \frac{\sqrt{x^2+y^2+(x^2+y^2)}}{\sqrt{x^2+y^2}} \, dA$$

$$= \delta \iint_R (x^2+y^2) \cdot \sqrt{2} \frac{\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}} \, dA$$

$$= \sqrt{2}\,\delta \int_0^{2\pi}\int_1^2 r^2 \cdot r \, dr\, d\theta = \sqrt{2}\,\delta \int_0^{2\pi}\int_1^2 r^3 \, dr\, d\theta$$

$$= \sqrt{2}\,\delta \int_0^{2\pi} \left(\frac{1}{4} r^4 \Big|_1^2 \right) d\theta$$

$$= \sqrt{2}\,\delta \int_0^{2\pi} \frac{15}{4} \, d\theta = \frac{15\sqrt{2}}{4} \delta \, \theta \Big|_0^{2\pi}$$

$$= \frac{15\sqrt{2}}{2} \delta \pi$$