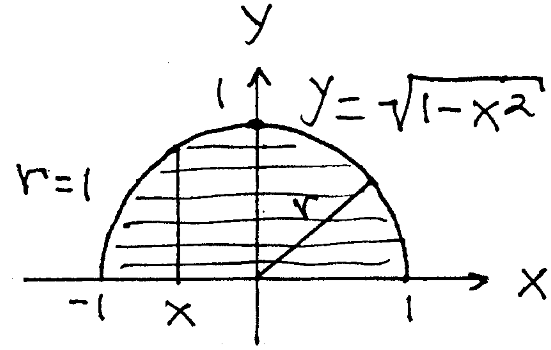
Section 15.4

9.) $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$

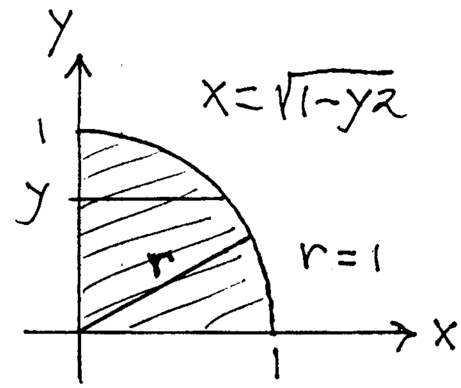


$= \int_0^{\pi}\int_0^1 r\,dr\,d\theta$

$= \int_0^{\pi}\left(\frac{1}{2}r^2\Big|_{r=0}^{r=1}\right)d\theta = \int_0^{\pi}\left(\frac{1}{2}(1)^2-\frac{1}{2}(0)^2\right)d\theta$

$= \int_0^{\pi}\frac{1}{2}\,d\theta = \frac{1}{2}\theta\Big|_0^{\pi} = \frac{1}{2}\pi-\frac{1}{2}(0) = \frac{1}{2}\pi$

10.) $\int_0^1\int_0^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$



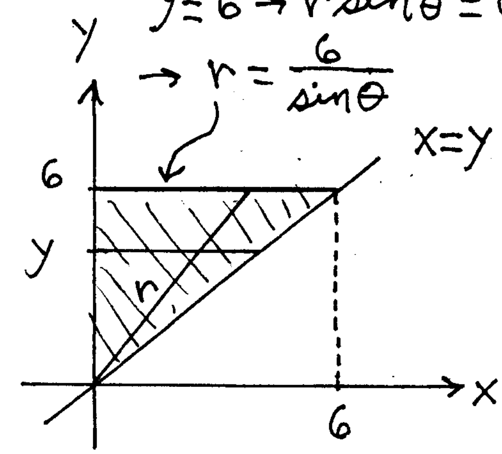
$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2\cdot r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\left(\frac{1}{4}r^4\Big|_{r=0}^{r=1}\right)d\theta$

$= \int_0^{\frac{\pi}{2}}\frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$

13) $\int_0^6 \int_0^y x \, dx \, dy$



$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{6}{\sin\theta}} r\cos\theta \cdot r \, dr \, d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{6}{\sin\theta}} r^2 \cos\theta \, dr \, d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \cdot \cos\theta \, \Big|_{r=0}^{r=\frac{6}{\sin\theta}} \right) d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}(6)^3 \cdot \frac{1}{\sin^3\theta} \cos\theta \, d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 72 \cdot \frac{\cos\theta}{\sin\theta} \cdot \left(\frac{1}{\sin\theta}\right)^2 d\theta$

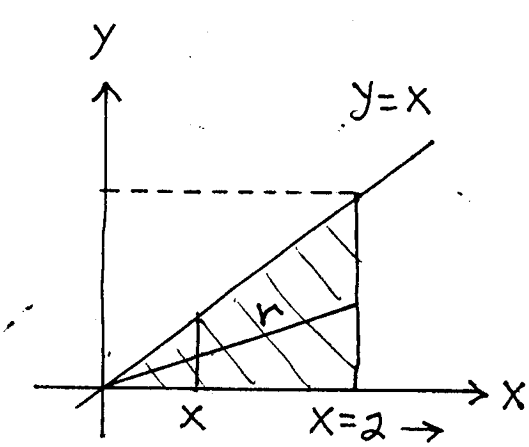
$= 72 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \cdot \csc^2\theta \, d\theta = -72 \cdot \frac{1}{2} \cot^2\theta \, \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

(Let $u = \cot\theta \to \ldots$)

$= -36 \cot^2\left(\frac{\pi}{2}\right) - -36 \cot^2\left(\frac{\pi}{4}\right)$

$= -36(0)^2 + 36(1)^2 = 36$

14.) $\int_0^2 \int_0^x y \, dy \, dx$



$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos\theta}} r\sin\theta \cdot r \, dr \, d\theta$

$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos\theta}} r^2 \sin\theta \, dr \, d\theta$

$r\cos\theta = 2 \to r = \frac{2}{\cos\theta}$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{3} r^3 \sin\theta \Big|_{r=0}^{r=\frac{2}{\cos\theta}} \right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}(2)^3 \cdot \frac{1}{\cos^3\theta} \sin\theta \, d\theta$$

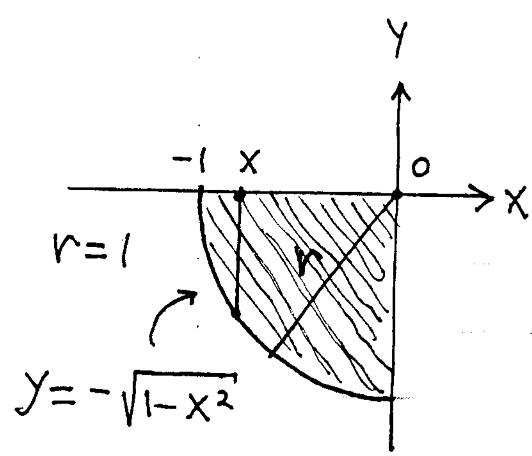
$$= \frac{8}{3} \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \cdot \left(\frac{1}{\cos\theta} \right)^2 d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{4}} \tan\theta \cdot \sec^2\theta \, d\theta = \frac{8}{3} \cdot \frac{1}{2} \tan^2\theta \Big|_0^{\frac{\pi}{4}}$$

(Let $u = \tan\theta \rightarrow \ldots$)

$$= \frac{4}{3} \tan^2\frac{\pi}{4} - \frac{4}{3} \cancelto{0}{\tan^2 0} = \frac{4}{3}(1)^2 = \frac{4}{3}$$

17) $\displaystyle\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}} \, dy \, dx$



$$= \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \frac{2}{1+\sqrt{r^2}} \cdot r \, dr \, d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \frac{2r}{1+r} \, dr \, d\theta$$

$$= 2 \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \frac{1+r-1}{1+r} \, dr \, d\theta$$

$$= 2 \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \left[\frac{1+r}{1+r} - \frac{1}{1+r} \right] dr \, d\theta$$

$$= 2 \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \left[1 - \frac{1}{1+r} \right] dr \, d\theta$$

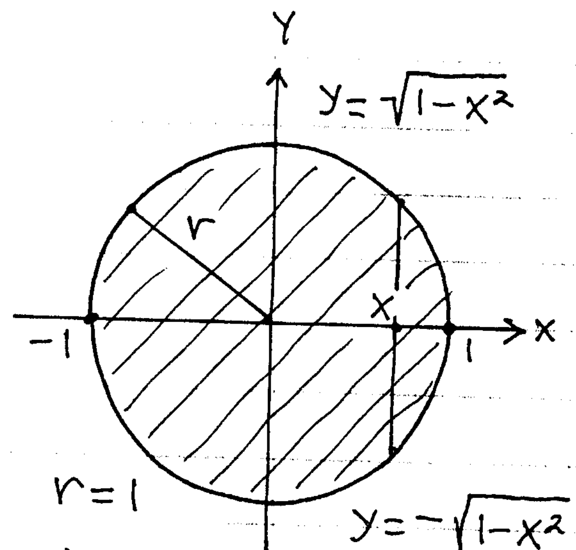
$$= 2 \int_{\pi}^{\frac{3\pi}{2}} \left(r - \ln|1+r| \right) \Big|_{r=0}^{r=1} d\theta$$

$$= 2\int_{\pi}^{\frac{3\pi}{2}} \left[(1-\ln 2) - (0 - \cancel{\ln 1}^{\,0})\right] d\theta$$

$$= 2(1-\ln 2)\cdot \theta \Big|_{\pi}^{3\pi/2} = 2(1-\ln 2)\left(\frac{3\pi}{2} - \frac{2\pi}{2}\right)$$

$$= \cancel{2}(1-\ln 2)\cdot \frac{1}{\cancel{2}}\pi = (1-\ln 2)\cdot \pi$$

18.) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\, dy\, dx$$



$$= \int_0^{2\pi}\int_0^1 \frac{2}{(1+r^2)^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{2r}{(1+r^2)^2}\, dr\, d\theta$$

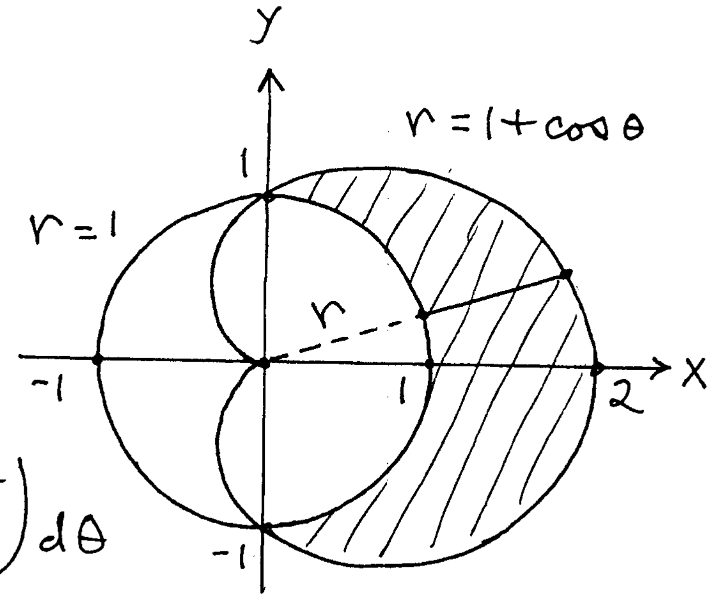
(Let $u = 1 + r^2 \to \ldots$)

$$= \int_0^{2\pi} \left(\frac{-1}{1+r^2}\Big|_{r=0}^{r=1}\right) d\theta = \int_0^{2\pi}\left(\frac{-1}{2} - \frac{-1}{1}\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \frac{1}{2}(2\pi) - \frac{1}{2}(0)$$

$$= \pi$$

28.) Area $= \iint_R 1 \, dA$



$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r \, dr \, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} r^2 \Big|_{r=1}^{r=1+\cos\theta} \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}(1+\cos\theta)^2 - \frac{1}{2}(1)^2 \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}(1 + 2\cos\theta + \cos^2\theta) - \frac{1}{2} \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cancel{\frac{1}{2}} + \cos\theta + \frac{1}{2}\cos^2\theta - \cancel{\frac{1}{2}} \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\theta + \frac{1}{2} \cdot \frac{1}{2}(1 + \cos 2\theta) \right) d\theta$

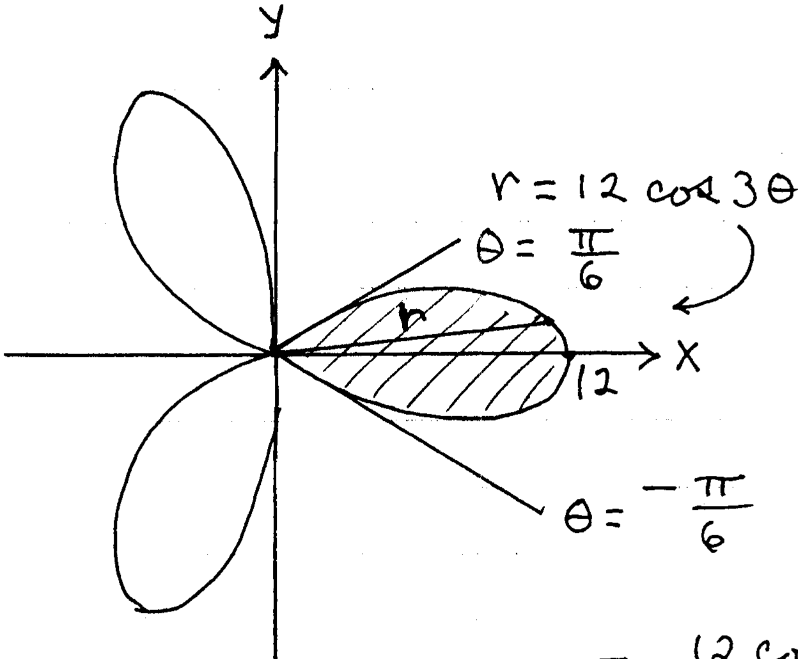
$= \left(\sin\theta + \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \left(\overset{1}{\sin\frac{\pi}{2}} + \frac{1}{4} \cdot \frac{\pi}{2} + \frac{1}{8} \overset{0}{\sin\pi} \right)$

$\quad - \left(\underset{-1}{\sin\left(-\frac{\pi}{2}\right)} + \frac{1}{4} \cdot -\frac{\pi}{2} + \frac{1}{8} \overset{0}{\sin(-\pi)} \right)$

$= 1 + \frac{\pi}{8} + 1 + \frac{\pi}{8} = 2 + \frac{\pi}{4}$

29.)



$\cos 3\theta = 0 \rightarrow$

$3\theta = \pm\frac{\pi}{2} \rightarrow$

$\theta = \pm\frac{\pi}{6}$

$$\text{Area} = \iint_R 1\, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{12\cos 3\theta} r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2} r^2 \Big|_{r=0}^{r=12\cos 3\theta} \right) d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(12)^2 \cos^2 3\theta \, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 72 \cdot \frac{1}{2}(1 + \cos 6\theta)\, d\theta$$

$$= 36\left(\theta + \frac{1}{6}\sin 6\theta\right) \Big|_{-\frac{\pi}{6}}^{\pi/6}$$

$$= 36\left(\frac{\pi}{6} + \frac{1}{6}\overset{0}{\cancel{\sin \pi}}\right) - 36\left(-\frac{\pi}{6} + \frac{1}{6}\overset{0}{\cancel{\sin(-\pi)}}\right)$$

$$= 12\pi$$

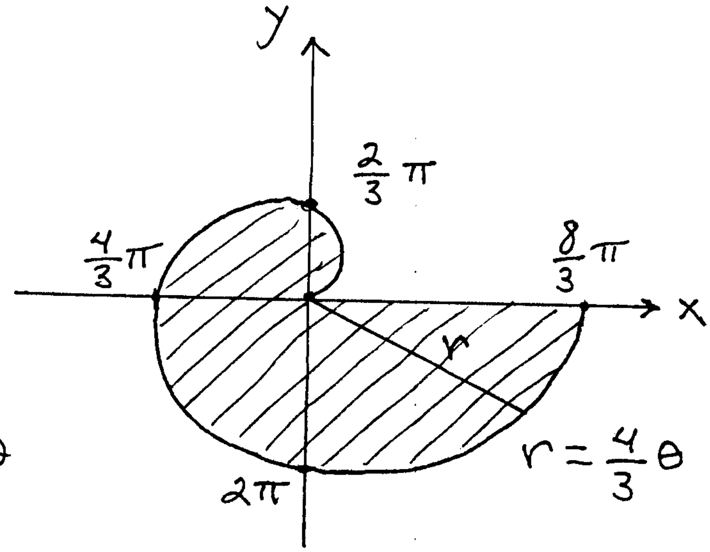
30.) $\text{Area} = \iint_R 1\, dA$

$$= \int_0^{2\pi} \int_0^{\frac{4}{3}\theta} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} r^2 \Big|_{r=0}^{r=\frac{4}{3}\theta} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left(\frac{4}{3}\theta \right)^2 d\theta$$

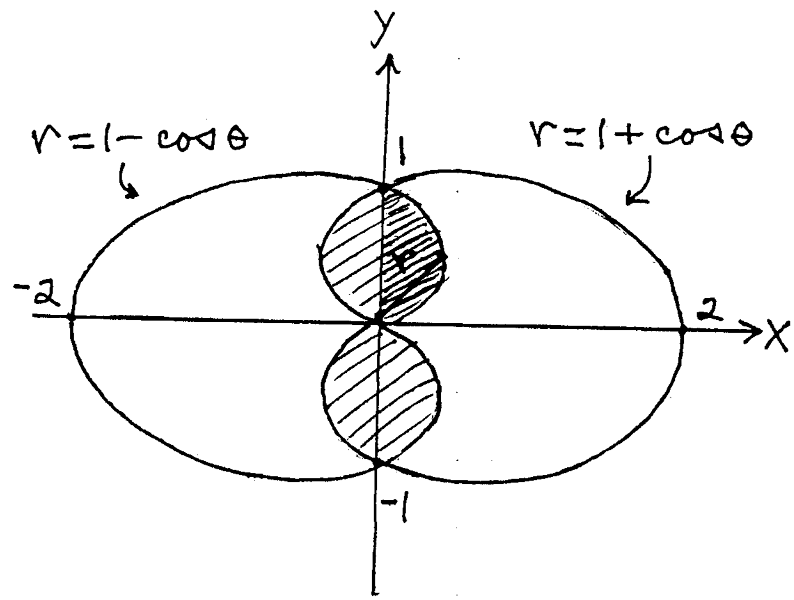
$$= \int_0^{2\pi} \frac{8}{9} \theta^2\, d\theta = \frac{8}{9} \cdot \frac{1}{3} \theta^3 \Big|_0^{2\pi} = \frac{8}{27} (2\pi)^3 = \frac{64}{27} \pi^3$$



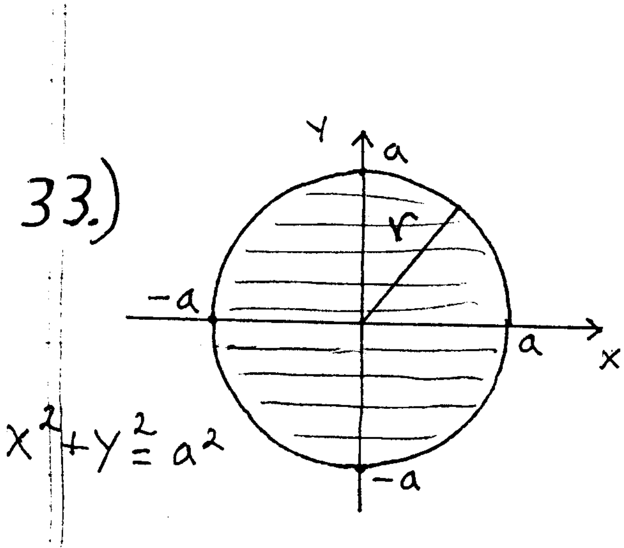
32.) By symmetry

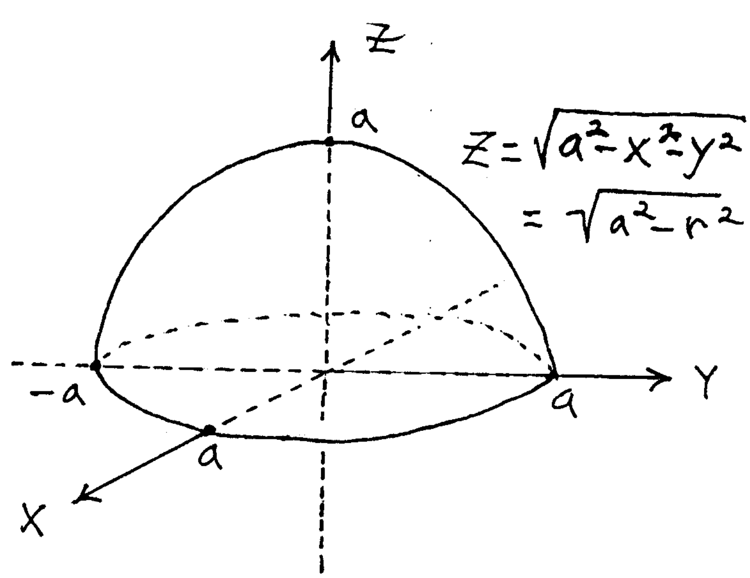
$\text{Area} = \iint_R 1\, dA$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^{1-\cos\theta} r\, dr\, d\theta$$



33.)





$$AVE = \frac{1}{\text{area } R} \iint_R f(P)\, dA$$

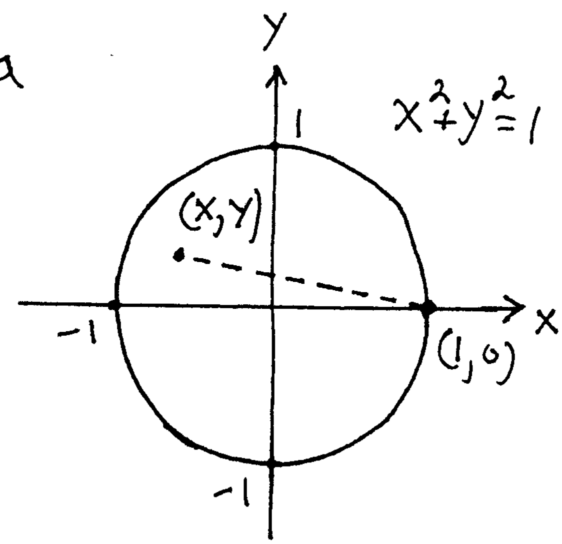
$$= \frac{1}{\pi a^2} \cdot \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \cdot r\, dr\, d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} -\frac{1}{3}(a^2 - r^2)^{3/2} \Big|_{r=0}^{r=a} \, d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \left[-\frac{1}{3}(0)^{3/2} - -\frac{1}{3}(a^2)^{3/2} \right] d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \frac{1}{3} a^3 \, d\theta = \frac{1}{\pi a^2} \cdot \frac{1}{3} a^3 \cdot \theta \Big|_0^{2\pi}$$

$$= \frac{1}{\pi} \cdot \frac{1}{3} a \cdot (2\pi - 0) = \frac{2}{3} a$$



36.) Distance squared:

$$f(x,y) = (x-1)^2 + (y-0)^2$$
$$= x^2 - 2x + 1 + y^2$$
$$= (x^2+y^2) - 2x + 1 \,,$$

$$AVE = \frac{1}{\text{area } R} \cdot \iint_R f(P)\, dA$$

$$= \frac{1}{\pi (1)^2} \cdot \int_0^{2\pi} \int_0^1 (r^2 - 2r\cos\theta + 1) \cdot r\, dr\, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r^3 - 2r^2 \cos\theta + r)\, dr\, d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{4}r^4 - \frac{2}{3}r^3\cos\theta + \frac{1}{2}r^2\right)\Big|_{r=0}^{r=1}\,d\theta$$

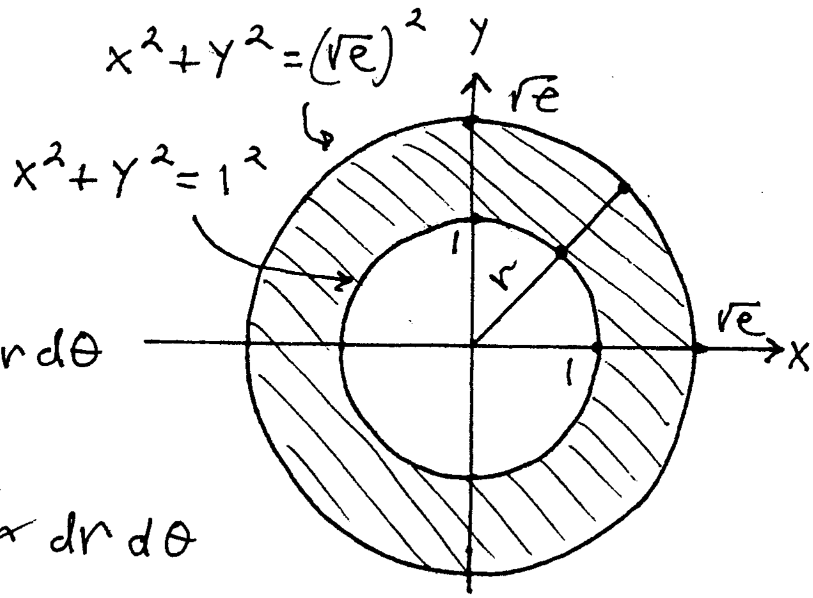
$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{4} - \frac{2}{3}\cos\theta + \frac{1}{2}\right)d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3}{4} - \frac{2}{3}\cos\theta\right)d\theta$$

$$= \frac{1}{\pi}\left(\frac{3}{4}\theta - \frac{2}{3}\sin\theta\right)\Big|_0^{2\pi}$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{3}{4}(2\cancel{\pi}) - \frac{2}{3}\overbrace{\cancel{\sin}\,2\pi}^{0}\right) - \frac{1}{\pi}\left(\frac{3}{4}(0) - \frac{2}{3}\overbrace{\cancel{\sin}\,0}^{0}\right)$$

$$= 3/2$$



37.) $\iint\limits_R f(P)\,dA$

$$= \int_0^{2\pi}\int_1^{\sqrt{e}} \frac{\ln r^2}{\sqrt{r^2}}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{e}} \frac{2\ln r}{\cancel{r}}\cdot\cancel{r}\,dr\,d\theta$$

(Let $u = \ln r$, $dv = dr$
$\rightarrow du = \frac{1}{r}dr$, $v = r$)

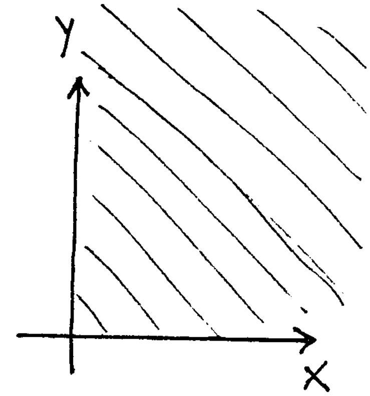
$$= \int_0^{2\pi}\left[r\ln r\Big|_{r=1}^{r=\sqrt{e}} - \int_1^{\sqrt{e}}\frac{1}{\cancel{r}}\cancel{r}\,dr\right]d\theta$$

$$= \int_0^{2\pi} \left[\left(\sqrt{e}\,\ln e^{1/2} - 1 \cdot \cancel{\ln 1}^{\,0} \right) - r \Big|_{r=1}^{r=\sqrt{e}} \right] d\theta$$

$$= \int_0^{2\pi} \left[\sqrt{e} \cdot \frac{1}{2} - (\sqrt{e} - 1) \right] d\theta$$

$$= \int_0^{2\pi} \left(1 - \frac{1}{2}\sqrt{e} \right) d\theta = \left(1 - \frac{1}{2}\sqrt{e} \right) \cdot \theta \Big|_0^{2\pi}$$

$$= \left(1 - \frac{1}{2}\sqrt{e} \right) 2\pi$$



41) a) $$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\lim_{A \to \infty} \int_0^{A} e^{-r^2} \cdot r\, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\lim_{A \to \infty} \frac{-1}{2} e^{-r^2} \Big|_{r=0}^{r=A} \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\lim_{A \to \infty} \left(\frac{-1}{2} e^{-A^2} - \frac{-1}{2} \cancel{e^0}^{\,1} \right) \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\lim_{A \to \infty} \left(\frac{-1}{2} \cdot \cancel{\frac{1}{e^{A^2}}}^{\,0} + \frac{1}{2} \right) \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\, d\theta = \frac{1}{2}\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{\pi}{4} \rightarrow I^2 = \frac{\pi}{4} \text{ so } I = \frac{1}{2}\sqrt{\pi}$$

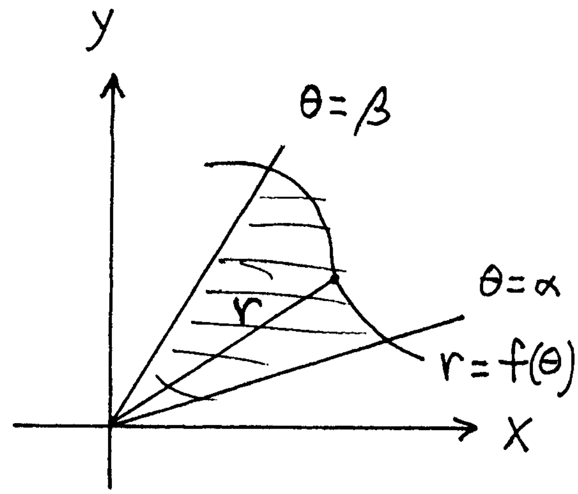
44) $\text{Area} = \iint_R 1\, dA$

$= \int_\alpha^\beta \int_0^{f(\theta)} r\, dr\, d\theta$

$= \int_\alpha^\beta \left(\frac{1}{2} r^2 \Big|_{r=0}^{r=f(\theta)} \right) d\theta$

$= \int_\alpha^\beta \left[\frac{1}{2}(f(\theta))^2 - \frac{1}{2}(0)^2 \right] d\theta$

$= \int_\alpha^\beta \frac{1}{2} (\underbrace{f(\theta)}_{r})^2\, d\theta$



46) $\text{Area} = \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \int_{\csc\theta}^{2\sin\theta} r\, dr\, d\theta$;

$\frac{\pi}{4} \le \theta \le \frac{3}{4}\pi$, $\csc\theta \le r \le 2\sin\theta$;

$r = \csc\theta \rightarrow r = \frac{1}{\sin\theta} \rightarrow r\sin\theta = 1 \rightarrow y = 1$;

$\csc\theta = 2\sin\theta \rightarrow$

$\frac{1}{\sin\theta} = 2\sin\theta \rightarrow$

$\sin^2\theta = \frac{1}{2} \rightarrow$

$\sin\theta = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\rightarrow \theta = \frac{\pi}{4}$, $\theta = \frac{3}{4}\pi$