Section 15.7

1) $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(z\Big|_{z=r}^{z=\sqrt{2-r^2}}\right) r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2}-r\right) r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2}-r^2\right) dr\,d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{3}r^3\right)\Big|_{r=0}^{r=1} d\theta$

$= \int_0^{2\pi} \left[\left(\frac{-1}{3}-\frac{1}{3}\right)-\left(\frac{-1}{3}\cdot 2^{3/2}-0\right)\right] d\theta$

$= \int_0^{2\pi} \left(-\frac{2}{3}+\frac{1}{3}2^{3/2}\right) d\theta = \left(\frac{-2}{3}+\frac{1}{3}\cdot 2\cdot\sqrt{2}\right)\theta\Big|_0^{2\pi}$

$= \left(\frac{-2}{3}+\frac{2}{3}\sqrt{2}\right)(2\pi-0) = \frac{4\pi}{3}(\sqrt{2}-1)$

6) $\int_0^{2\pi}\int_0^1\int_{-\frac{1}{2}}^{\frac{1}{2}} (r^2\sin^2\theta + z^2)\,dz\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(r^2\sin^2\theta\cdot z+\frac{1}{3}z^3\right)\Big|_{z=-\frac{1}{2}}^{z=\frac{1}{2}}\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \Big[\left(r^2\sin^2\theta\cdot\frac{1}{2}+\frac{1}{3}\left(\frac{1}{8}\right)\right)$
$\qquad - \left(r^2\sin^2\theta\left(\frac{-1}{2}\right)+\frac{1}{3}\left(\frac{-1}{8}\right)\right)\Big]\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{12}+r^2\sin^2\theta\right)\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{12}r+r^3\sin^2\theta\right) dr\,d\theta$

$$= \int_0^{2\pi} \left(\frac{1}{24} r^2 + \frac{1}{4} r^4 \sin^2\theta \right) \Big|_{r=0}^{r=1} \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{4} \sin^2\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{4} \cdot \frac{1}{2} (1 - \cos 2\theta) \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{8} - \frac{1}{8} \cos 2\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{6} - \frac{1}{8} \cos 2\theta \right) d\theta$$

$$= \left(\frac{1}{6}\theta - \frac{1}{8} \cdot \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi}$$

$$= \left(\frac{2\pi}{6} - \frac{1}{16} \cancelto{0}{\sin 4\pi} \right) - \left(0 - \frac{1}{16} \cancelto{0}{\sin 0} \right) = \frac{\pi}{3}$$

8.) $$\int_{-1}^{1} \int_0^{2\pi} \int_0^{1+\cos\theta} 4r \, dr \, d\theta \, dz$$

$$= \int_{-1}^{1} \int_0^{2\pi} \left(2r^2 \Big|_{r=0}^{r=1+\cos\theta} \right) d\theta \, dz$$

$$= \int_{-1}^{1} \int_0^{2\pi} 2(1+\cos\theta)^2 \, d\theta \, dz$$

$$= \int_{-1}^{1} \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \, dz$$

$$= \int_{-1}^{1} \int_0^{2\pi} 2\left(1 + 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right) d\theta \, dz$$

$$= \int_{-1}^{1} \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \, dz$$

$$= \int_{-1}^{1} \left(3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta\right) \Big|_{\theta=0}^{\theta=2\pi} dz$$

$$= \int_{-1}^{1} \Big[\left(6\pi + 4 \cancelto{0}{\sin 2\pi} + \frac{1}{2} \cancelto{0}{\sin(4\pi)}\right) - \left(0 + 4 \cancelto{0}{\sin 0} + \frac{1}{2} \cancelto{0}{\sin 0}\right) \Big] dz$$

$$= \int_{-1}^{1} 6\pi \, dz = 6\pi \cdot z \Big|_{-1}^{1} = 6\pi(1 - -1)$$

$$= 12\pi$$

10.) $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1) r \, d\theta \, dz \, dr$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r^2 \sin\theta + r) \, d\theta \, dz \, dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} (-r^2\cos\theta + r\theta) \Big|_{\theta=0}^{\theta=2\pi} dz \, dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left[(-r^2 \overset{1}{\cancel{\cos 2\pi}} + r \cdot 2\pi) - (-r^2 \overset{1}{\cancel{\cos 0}} + r \cdot 0) \right] dz \, dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} (\cancel{-r^2} + 2\pi r + \cancel{r^2}) \, dz \, dr$$

$$= \int_0^2 \left(2\pi r \cdot z \Big|_{z=r-2}^{z=\sqrt{4-r^2}} \right) dr$$

$$= \int_0^2 2\pi r \left(\sqrt{4-r^2} - (r-2) \right) dr$$

$$= 2\pi \int_0^2 (r \cdot \sqrt{4-r^2} - r^2 + 2r) \, dr$$

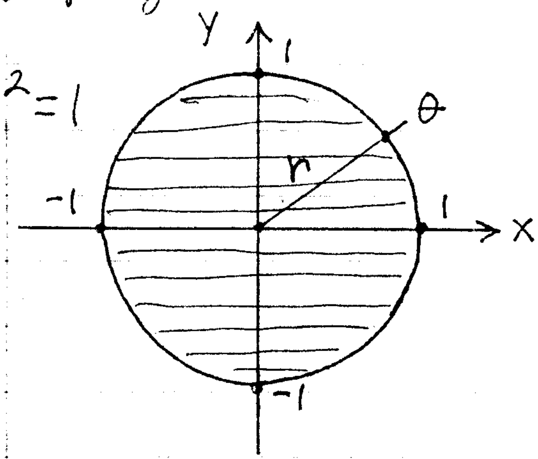
$$= 2\pi \cdot \left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{3}r^3 + r^2 \right) \Big|_0^2$$

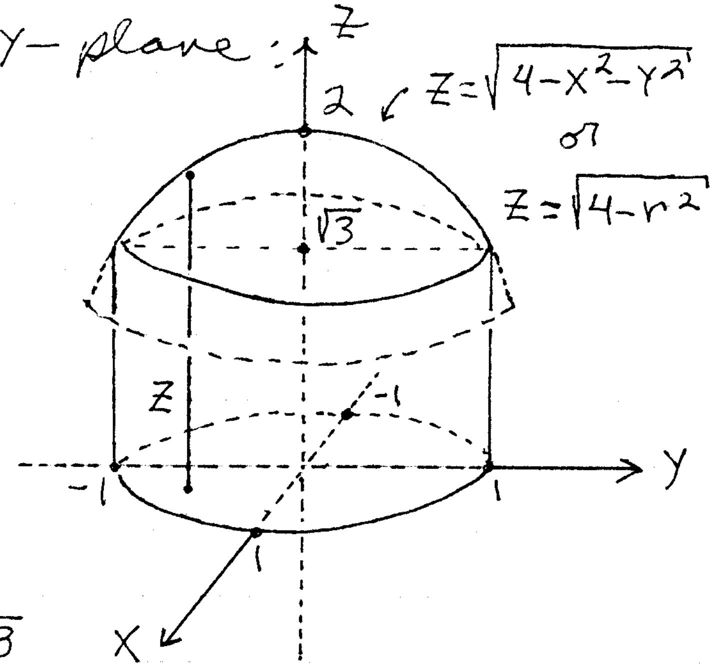
$$= 2\pi \left[\left(0 - \frac{8}{3} + 4\right) - \left(-\frac{1}{3}(4)^{3/2} - 0 + 0\right) \right]$$

$$= 2\pi \left[\cancel{\frac{-8}{3}} + \frac{12}{3} + \cancel{\frac{8}{3}} \right] = 8\pi$$

11.) projection onto xy-plane:

$x^2+y^2=1$

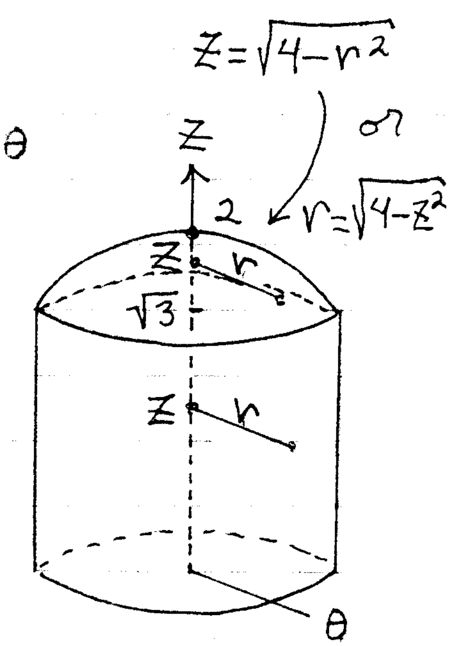




$x^2+y^2+z^2=4,\ x^2+y^2=1 \rightarrow$

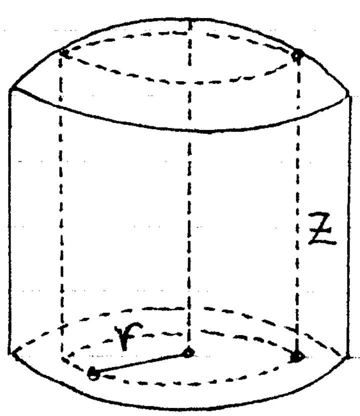
$1+z^2=4 \rightarrow z^2=3 \rightarrow z=\sqrt{3}$

a.) $\text{Vol} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$



b.) $\text{Vol} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r \, dr \, dz \, d\theta$

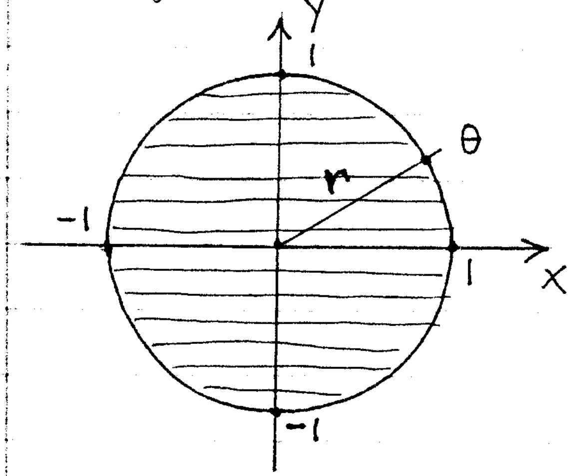
$+ \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$



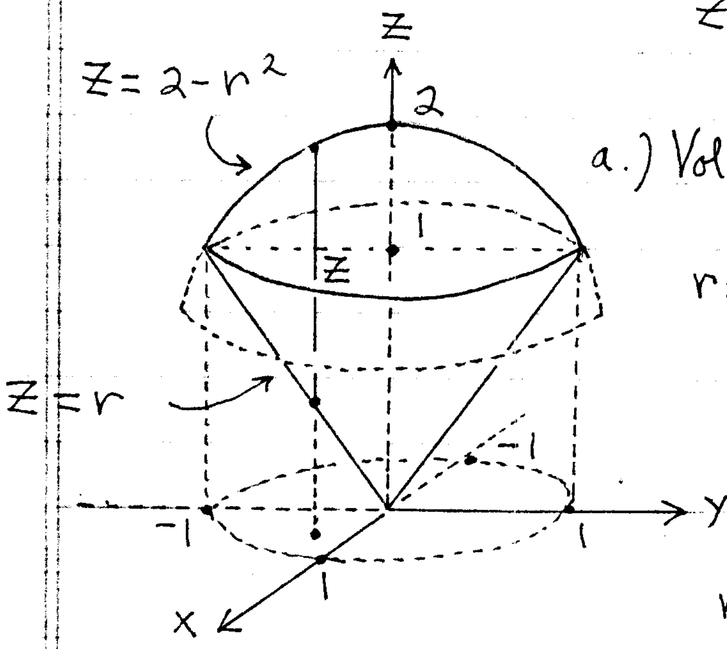
c.)

$\text{Vol} = \int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r \, d\theta \, dz \, dr$

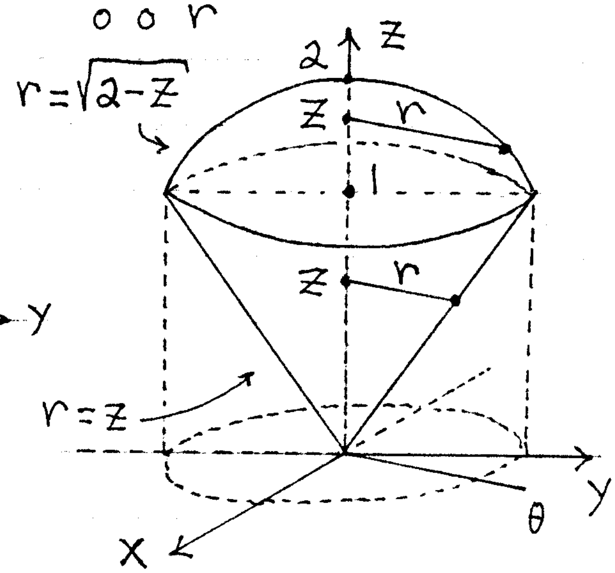
12.) projection onto xy-plane:



$$z=\sqrt{x^2+y^2} \text{ and}$$
$$z=2-x^2-y^2 \to$$
$$x^2+y^2=2-z \to$$
$$z=\sqrt{2-z} \to$$
$$z^2=2-z \to$$
$$z^2+z-2=0 \to$$
$$(z-1)(z+2)=0 \to$$
$$z=1 \qquad \cancel{z=-2} \to x^2+y^2=1$$



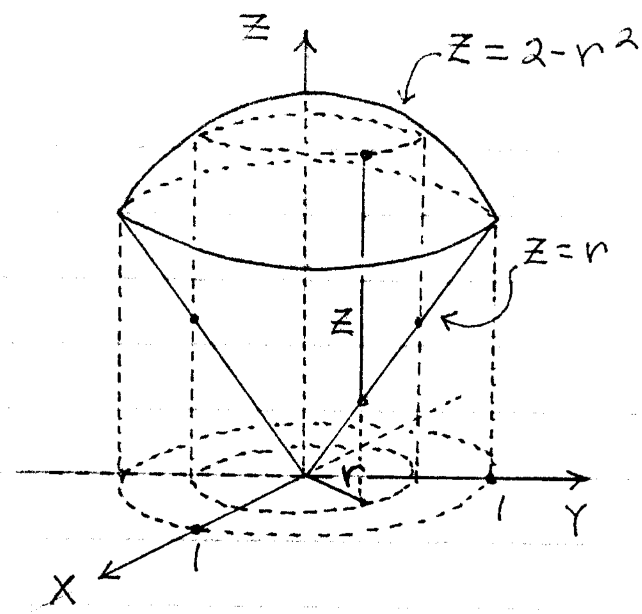
a.) $$\text{Vol}=\int_0^{2\pi}\int_0^{1}\int_r^{2-r^2} r\,dz\,dr\,d\theta$$



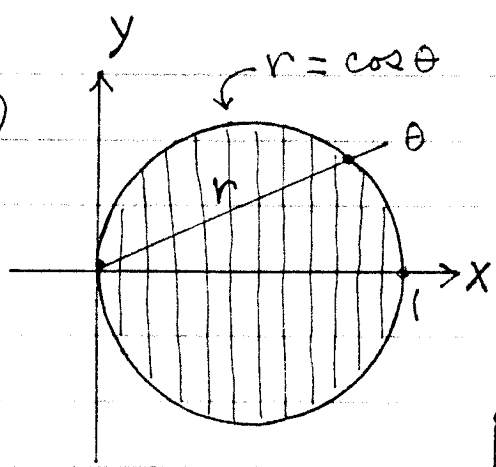
b.) $$\text{Vol}=\int_0^{2\pi}\int_0^{1}\int_0^{z} r\,dr\,dz\,d\theta$$
$$+\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta$$

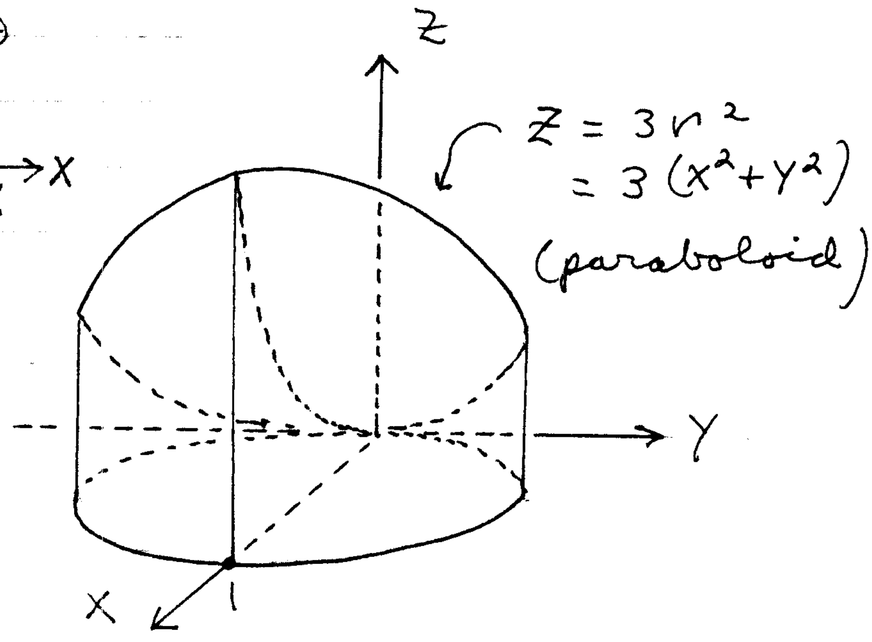
c.)

$$Vol = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr$$



13.)

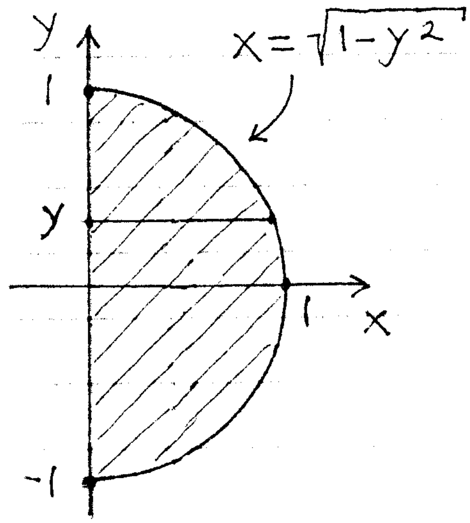


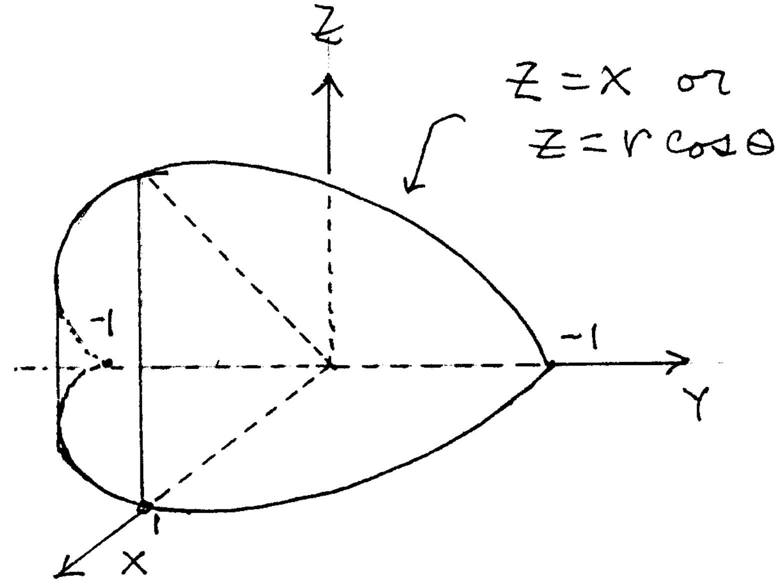


$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{3r^2} f(r,\theta,z)\cdot r \; dz \, dr \, d\theta$$

14.) $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x} (x^2+y^2)\, dz\, dx\, dy$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2 \cdot r\, dz\, dr\, d\theta$





$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3\, dz\, dr\, d\theta$

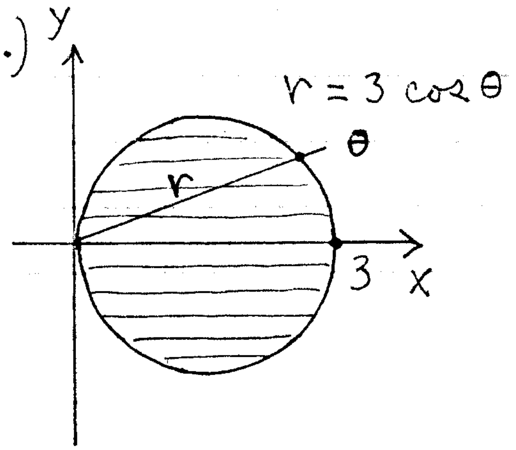
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} \left(r^3 \cdot z \Big|_{z=0}^{z=r\cos\theta} \right) dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^4\cos\theta\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{5} r^5 \cos\theta \Big|_{r=0}^{r=1} \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cos\theta\, d\theta = \frac{1}{5}\sin\theta \Big|_{-\frac{\pi}{2}}^{\pi/2} = \frac{1}{5}\sin\frac{\pi}{2} - \frac{1}{5}\sin\left(-\frac{\pi}{2}\right)$

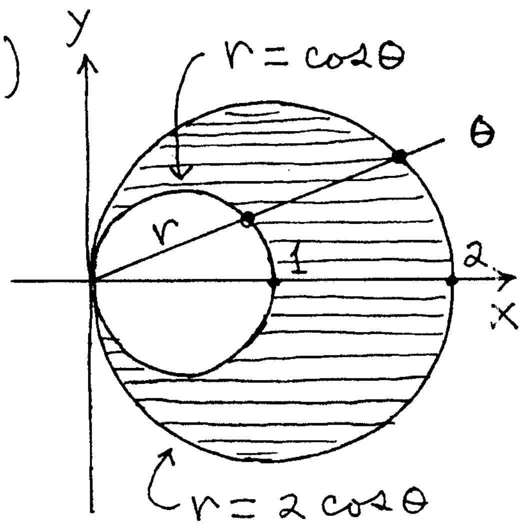
$= \frac{1}{5}(1) - \frac{1}{5}(-1) = \frac{2}{5}$

16.)



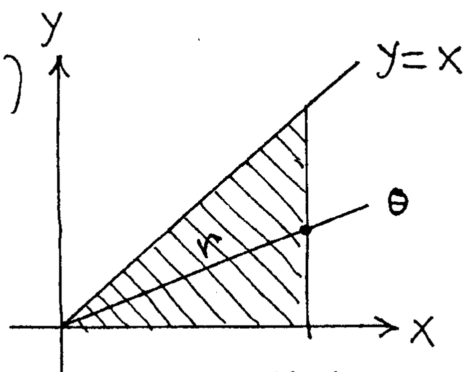
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{3\cos\theta} \int_{0}^{5 - r\cos\theta} f(r,\theta,z)\cdot r \, dz\, dr\, d\theta$$

18.)



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\cos\theta}^{2\cos\theta}\int_{0}^{3-r\sin\theta} f(r,\theta,z)\cdot r\,dz\,dr\,d\theta$$

19.)



$x=1$ or $r=\sec\theta$

$$\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta}\int_{0}^{2-r\sin\theta} f(r,\theta,z)\cdot r\,dz\,dr\,d\theta$$

43.) $\left.\begin{array}{l} z = 4-4(x^2+y^2) \\ z = (x^2+y^2)^2-1 \end{array}\right\} \rightarrow 4(x^2+y^2) = 4-z \rightarrow$

$x^2+y^2 = 1-\frac{1}{4}z \rightarrow$

(SUB) $z = (1-\frac{1}{4}z)^2 - 1 \rightarrow z = \cancel{1} - \frac{1}{2}z + \frac{1}{16}z^2 - \cancel{1} \rightarrow$

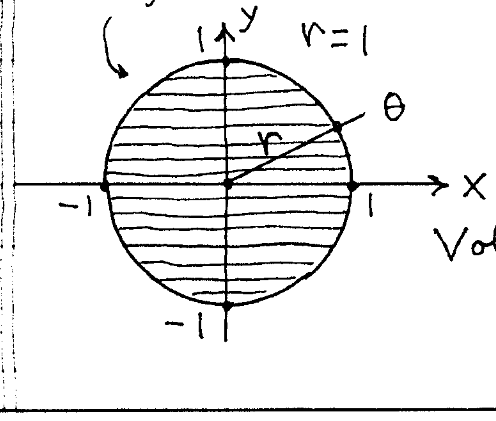
$16z = -8z + z^2 \rightarrow z^2 - 24z = 0 \rightarrow z(z-24) = 0$

$\rightarrow z = 0$ or $\cancel{z=24}$; $0 = 4-4(x^2+y^2) \rightarrow$

$x^2+y^2 = 1$ or $r=1$

$z = 4-4(x^2+y^2)$ or $z = 4-4r^2$,

$z = (x^2+y^2)^2-1$ or $z = r^4-1$:

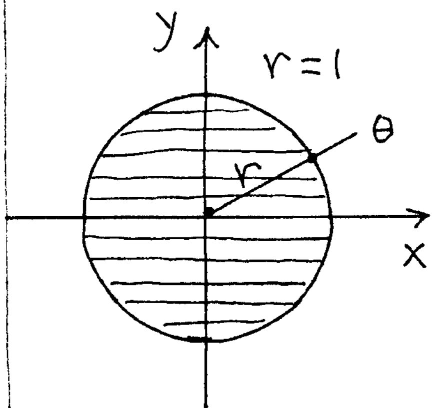


$$\text{Vol} = \int_{0}^{2\pi}\int_{0}^{1}\int_{r^4-1}^{4-4r^2} r\,dz\,dr\,d\theta$$

44.) $\left.\begin{array}{l} z = 1-r \\ z = -\sqrt{1-r^2} \end{array}\right\} \to 1-r = -\sqrt{1-r^2} \to (1-r)^2 = 1-r^2 \to$

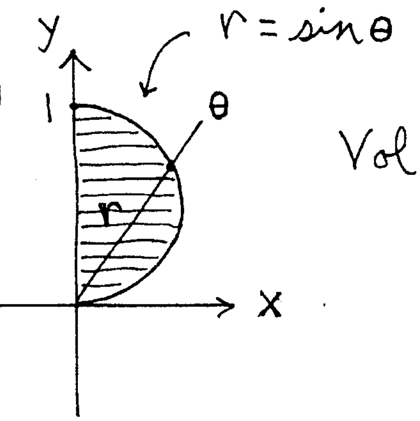
$\cancel{1} - 2r + r^2 = \cancel{1} - r^2 \to 2r^2 - 2r = 0 \to$

$2r(r-1) = 0 \to \cancel{r=0}$ or $r = 1$ and $z = 0$:

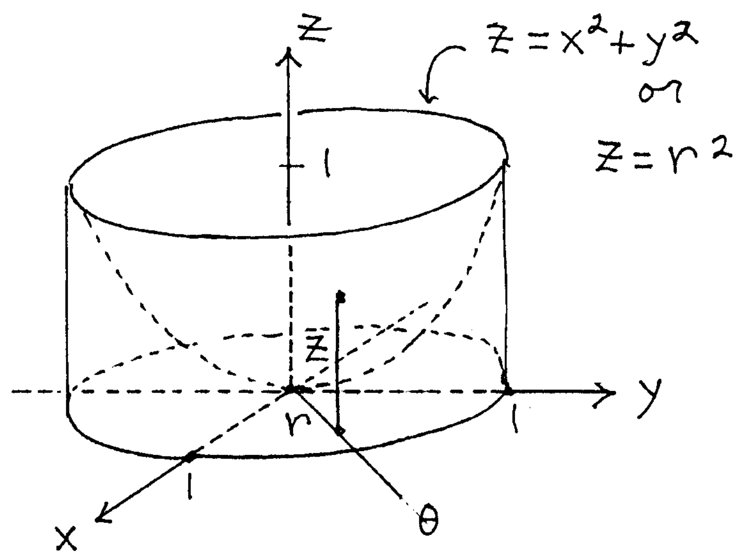


$$Vol = \int_0^{2\pi} \int_0^{1} \int_{-\sqrt{1-r^2}}^{1-r} r \, dz \, dr \, d\theta$$

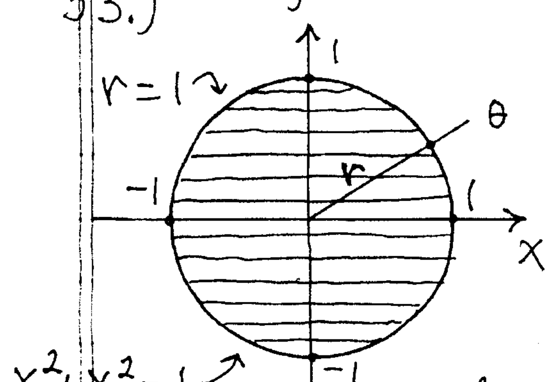
47.)



$$Vol = \int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

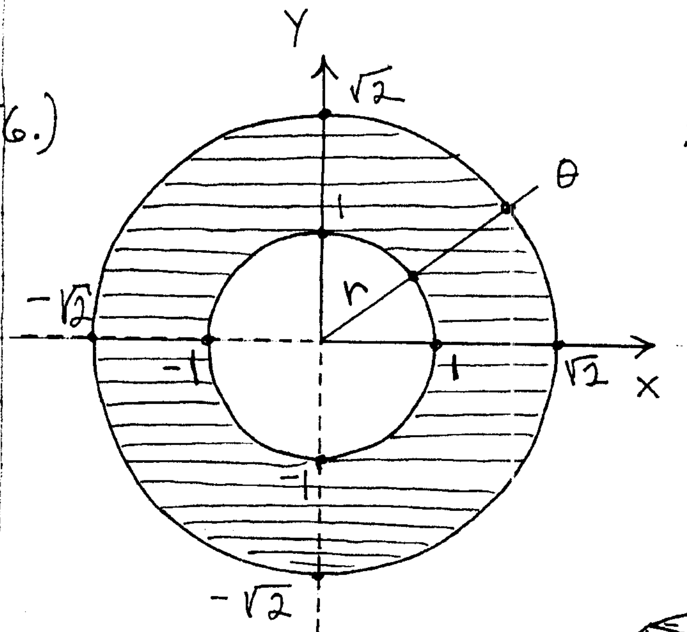


53.)



$$Vol = \int_0^{2\pi} \int_0^{1} \int_0^{r^2} r \, dz \, dr \, d\theta$$

56.)



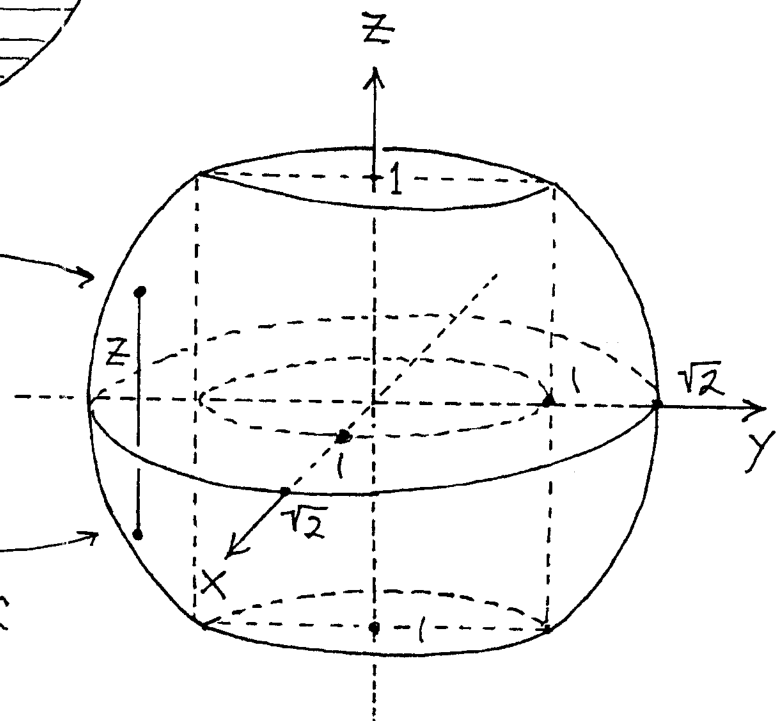
$$\left.\begin{array}{l} x^2+y^2+z^2=2 \\ x^2+y^2=1 \end{array}\right\} \to$$

$1+z^2=2 \to z^2=1 \to$

$z=\pm 1$;

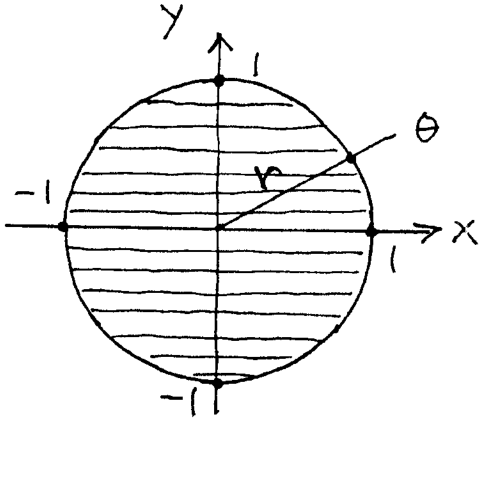
$z=\sqrt{2-x^2-y^2}$
or
$z=\sqrt{2-r^2}$

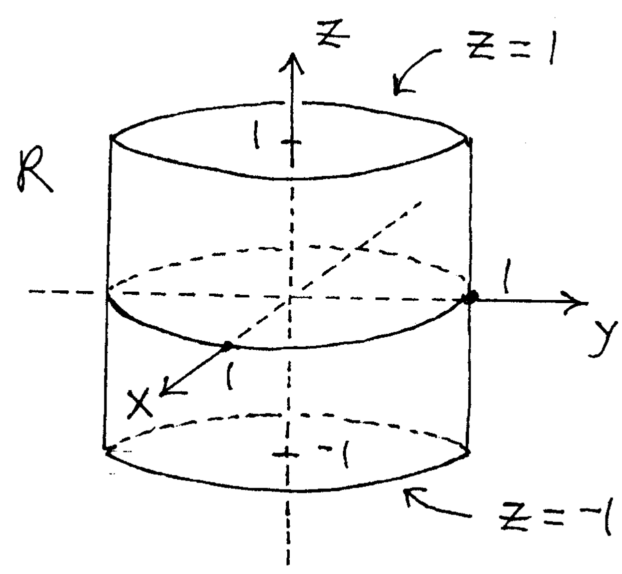
$z=-\sqrt{2-x^2-y^2}$
or $z=-\sqrt{2-r^2}$



$$Vol=\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r\ dz\ dr\ d\theta$$

63.)



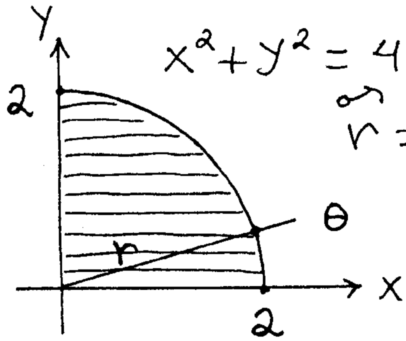


$$\text{AVE} = \frac{1}{\text{vol. } R} \cdot \iiint_R f(P)\, dV$$

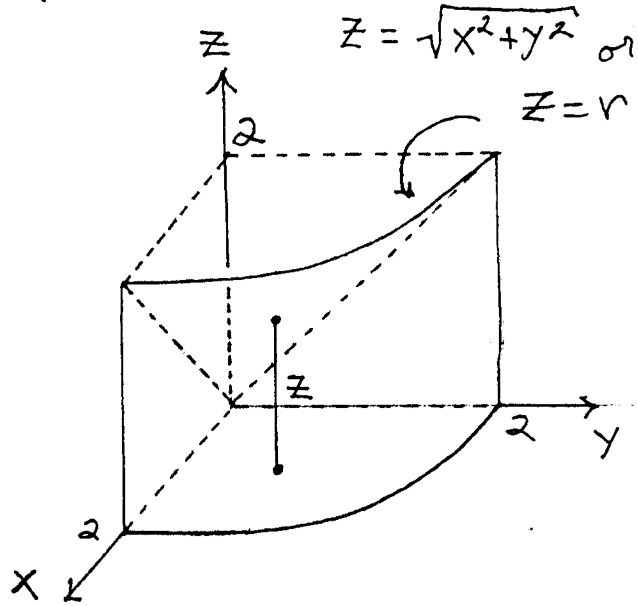
$$= \frac{1}{\pi(1)^2(2)} \int_0^{2\pi}\int_0^1\int_{-1}^1 r \cdot r\, dz\, dr\, d\theta$$

68.)

$x^2+y^2=4$ or $r=2$



$z=\sqrt{x^2+y^2}$ or $z=r$



$$\left.\begin{array}{l} z=\sqrt{x^2+y^2} \\ x^2+y^2=4 \end{array}\right\} \rightarrow$$

$$z=\sqrt{4}=2$$

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r (r\cos\theta)\cdot r\, dz\, dr\, d\theta}{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r r\, dz\, dr\, d\theta},$$

$$\bar{y} = \frac{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r (r\sin\theta)\cdot r\, dz\, dr\, d\theta}{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r r\, dz\, dr\, d\theta},$$

$$\bar{z} = \frac{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r z\cdot r\, dz\, dr\, d\theta}{\int_0^{\frac{\pi}{2}}\int_0^2\int_0^r r\, dz\, dr\, d\theta}$$