

# Section 15.7

$$\begin{aligned}
 21.) & \int_0^\pi \int_0^\pi \int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \left( \frac{1}{3} \rho^3 \sin\phi \Big|_{\rho=0}^{\rho=2\sin\phi} \right) d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{1}{3} (8 \sin^3\phi) \sin\phi \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{8}{3} \sin^4\phi \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{8}{3} (\sin^2\phi)^2 \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{8}{3} \left( \frac{1}{2} (1 - \cos 2\phi) \right)^2 \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{8}{3} \cdot \frac{1}{4} (1 - 2\cos 2\phi + \cos^2 2\phi) \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{2}{3} (1 - 2\cos 2\phi + \frac{1}{2} (1 + \cos 4\phi)) \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^\pi \frac{2}{3} \left( \frac{3}{2} - 2\cos 2\phi + \frac{1}{2} \cos 4\phi \right) \, d\phi \, d\theta \\
 &= \int_0^\pi \frac{2}{3} \left( \frac{3}{2} \phi - 2 \cdot \frac{1}{2} \sin 2\phi + \frac{1}{2} \cdot \frac{1}{4} \sin 4\phi \right) \Big|_{\phi=0}^{\phi=\pi} d\theta \\
 &= \int_0^\pi \frac{2}{3} \left( \frac{3}{2} \pi - \cancel{\sin 2\pi} + \frac{1}{8} \cancel{\sin 4\pi} \right) - \frac{1}{3} (0) \, d\theta \\
 &= \int_0^\pi \pi \, d\theta = \pi \theta \Big|_0^\pi = \pi(\pi - 0) = \pi^2
 \end{aligned}$$

$$24.) \int_0^{\frac{3}{2}\pi} \int_0^\pi \int_0^1 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta.$$

$$\begin{aligned}
&= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \left( \frac{5}{4} \ell^4 \sin^3 \phi \Big|_{\ell=0}^{\ell=1} \right) d\phi d\theta \\
&= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin^3 \phi d\phi d\theta \\
&= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin \phi \cdot \sin^2 \phi d\phi d\theta \\
&= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin \phi (1 - \cos^2 \phi) d\phi d\theta \\
&= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} (\sin \phi - \sin \phi \cos^2 \phi) d\phi d\theta \\
&= \int_0^{\frac{3}{2}\pi} \frac{5}{4} \left( -\cos \phi + \frac{1}{3} \cos^3 \phi \right) \Big|_{\phi=0}^{\phi=\pi} d\theta \\
&= \int_0^{\frac{3}{2}\pi} \left[ \frac{5}{4} (-\cos \pi + \frac{1}{3} \cos^3 \pi) - \frac{5}{4} (-\cos 0 + \frac{1}{3} \cos^3 0) \right] d\theta \\
&= \int_0^{\frac{3}{2}\pi} \left[ \frac{5}{4} (-(-1) + \frac{1}{3}(-1)) - \frac{5}{4} (-1 + \frac{1}{3}(1)) \right] d\theta \\
&= \int_0^{\frac{3}{2}\pi} \left( \frac{5}{4} \cdot \frac{2}{3} - \frac{5}{4} \cdot \frac{-2}{3} \right) d\theta \\
&= \int_0^{\frac{3}{2}\pi} \frac{10}{6} d\theta = \int_0^{\frac{3}{2}\pi} \frac{5}{3} d\theta \\
&= \frac{5}{3} \theta \Big|_0^{\frac{3}{2}\pi} = \frac{5}{3} \cdot \frac{3}{2} \pi = \frac{5}{2} \pi
\end{aligned}$$

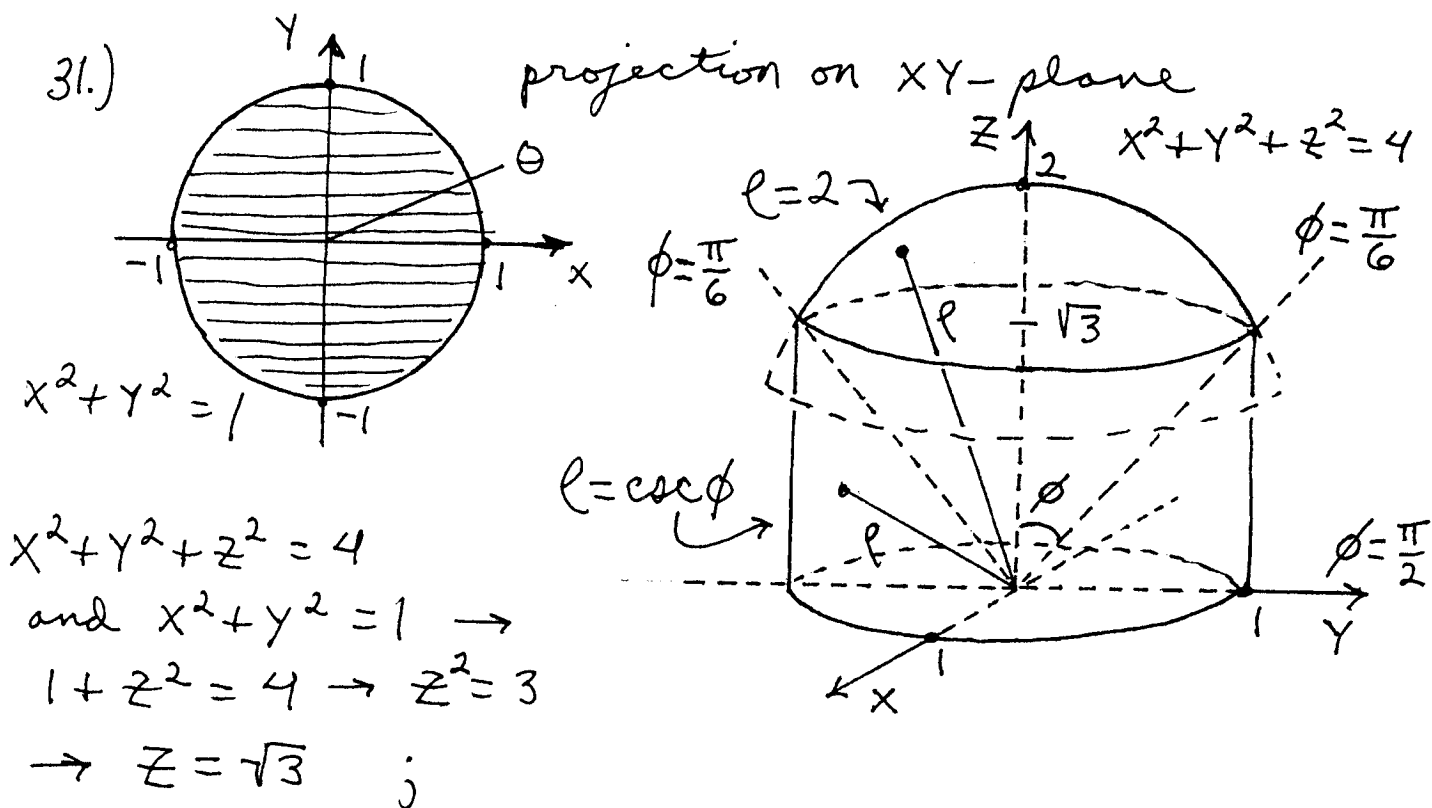
$$\begin{aligned}
26.) & \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec \phi} \ell^3 \cos \phi \sin \phi d\ell d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left( \frac{1}{4} \ell^4 \cdot \cos \phi \sin \phi \Big|_{\ell=0}^{\ell=\sec \phi} \right) d\phi d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^4 \phi \cos 2\phi \sin \phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^2 \phi \cdot \frac{1}{\cancel{\cos \phi}} \cdot \frac{1}{\cancel{\cos \phi}} \sin \phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^2 \phi \tan \phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \left( \frac{1}{4} \cdot \frac{1}{2} \tan^2 \phi \bigg|_{\phi=0}^{\phi=\frac{\pi}{4}} \right) d\theta \\
&= \int_0^{2\pi} \left( \frac{1}{8} \tan^2 \frac{\pi}{4} - \frac{1}{8} \cancel{\tan^2 0} \right) d\theta \\
&= \int_0^{2\pi} \frac{1}{8} (1)^2 \, d\theta = \frac{1}{8} \theta \bigg|_0^{2\pi} = \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
27.) \int_0^2 \int_{-\pi}^0 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^3 \sin 2\phi \, d\phi \, d\theta \, de \\
&= \int_0^2 \int_{-\pi}^0 \left( e^3 \cdot \frac{-1}{2} \cos 2\phi \bigg|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} \right) d\theta \, de \\
&= \int_0^2 \int_{-\pi}^0 \left( \frac{-1}{2} e^3 \cancel{\cos \pi} - \frac{-1}{2} e^3 \cancel{\cos \frac{\pi}{2}} \right) d\theta \, de \\
&= \int_0^2 \int_{-\pi}^0 \frac{1}{2} e^3 \, d\theta \, de = \int_0^2 \left( \frac{1}{2} e^3 \cdot \theta \bigg|_{\theta=-\pi}^{\theta=0} \right) de \\
&= \int_0^2 \left( \frac{1}{2} e^3 \cdot (0) - \frac{1}{2} e^3 (-\pi) \right) de \\
&= \int_0^2 \frac{\pi}{2} e^3 \, de = \frac{\pi}{2} \cdot \frac{1}{4} e^4 \bigg|_0^2 \\
&= \frac{\pi}{2} \cdot \frac{1}{4} \cdot 16 = 2\pi
\end{aligned}$$

$$\begin{aligned}
30.) & \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\csc \phi}^2 5 \ell^4 \sin^3 \phi \, d\ell \, d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \ell^5 \sin^3 \phi \Big|_{\ell=\csc \phi}^{\ell=2} \right) d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (32 \sin^3 \phi - \csc^5 \phi \cdot \sin^3 \phi) d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 32 \sin^3 \phi - \csc^2 \phi \cdot \frac{1}{\sin^3 \phi} \cdot \sin^3 \phi \right) d\theta \, d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (32 \sin^3 \phi - \csc^2 \phi) \cdot \theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (32 \sin^3 \phi - \csc^2 \phi) \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi (32 \sin \phi \cdot \sin^2 \phi - \csc^2 \phi) d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi (32 \sin \phi (1 - \cos^2 \phi) - \csc^2 \phi) d\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi (32 \sin \phi - 32 \sin \phi \cos^2 \phi - \csc^2 \phi) d\phi \\
&= \pi \left( -32 \cos \phi + 32 \cdot \frac{1}{3} \cos^3 \phi + \cot \phi \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \pi \left( -32 \cancel{\cos \frac{\pi}{2}} + \frac{32}{3} \cancel{\cos^3 \frac{\pi}{2}} + \cancel{\cos \frac{\pi}{2}} \right) \\
&\quad - \pi \left( -32 \cos \frac{\pi}{6} + \frac{32}{3} \cos^3 \frac{\pi}{6} + \cos \frac{\pi}{6} \right) \\
&= -\pi \left( -32 \cdot \frac{\sqrt{3}}{2} + \frac{32}{3} \cdot \left( \frac{\sqrt{3}}{2} \right)^3 + \frac{\sqrt{3}/2}{1/2} \right) \\
&= -\pi \left( -16\sqrt{3} + \frac{32}{3} \cdot \frac{3\sqrt{3}}{8} + \sqrt{3} \right) \\
&= -\pi \left( -11\sqrt{3} \right) = 11\sqrt{3} \cdot \pi
\end{aligned}$$



$$x^2 + y^2 = 1 \rightarrow r = 1 \rightarrow \ell \sin \phi = 1 \rightarrow \ell = \csc \phi$$

$$\begin{aligned}
\text{a.) Vol} &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 1 \cdot \ell^2 \sin \phi \, d\ell \, d\phi \, d\theta \\
&\quad + \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\csc \phi} 1 \cdot \ell^2 \sin \phi \, d\ell \, d\phi \, d\theta
\end{aligned}$$

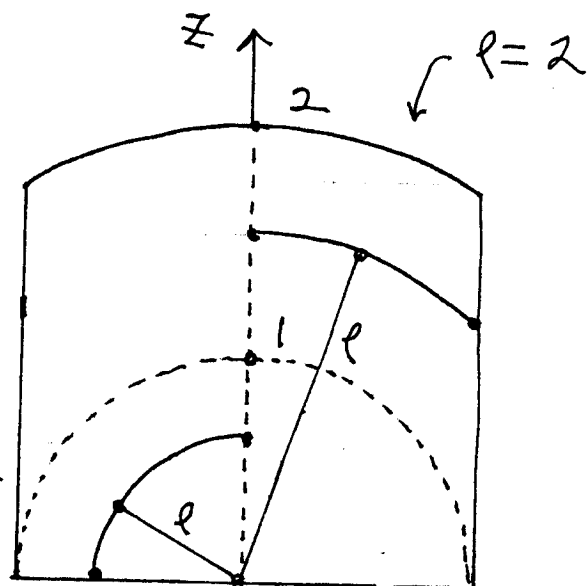
b.) SIDE VIEW  
OF SOLID

$$r=1 \rightarrow$$

$$\ell \sin \phi = 1 \rightarrow$$

$$\sin \phi = \frac{1}{\ell} \rightarrow$$

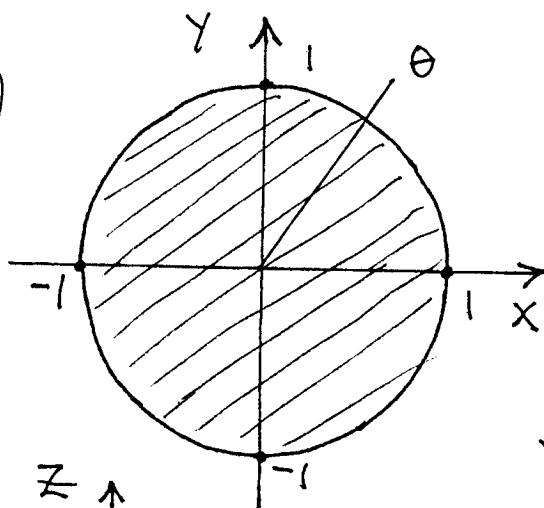
$$\phi = \arcsin\left(\frac{1}{\ell}\right)$$



$$\text{Vol} = \int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{2}} 1 \cdot \ell^2 \sin \phi \, d\phi \, d\ell \, d\theta$$

$$+ \int_0^{2\pi} \int_1^2 \int_0^{\arcsin(1/\ell)} 1 \cdot \ell^2 \sin \phi \, d\phi \, d\ell \, d\theta$$

32.)

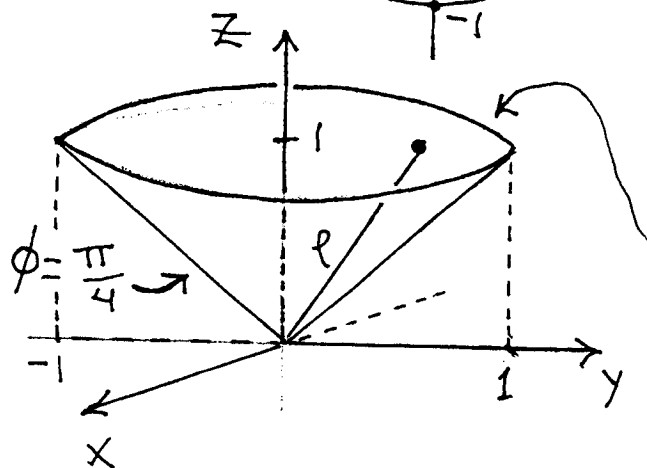


$$z = \sqrt{x^2 + y^2} \text{ and } z=1$$

$$\rightarrow 1 = \sqrt{x^2 + y^2} \rightarrow$$

$$x^2 + y^2 = 1 ;$$

projection on  
xy-plane ;



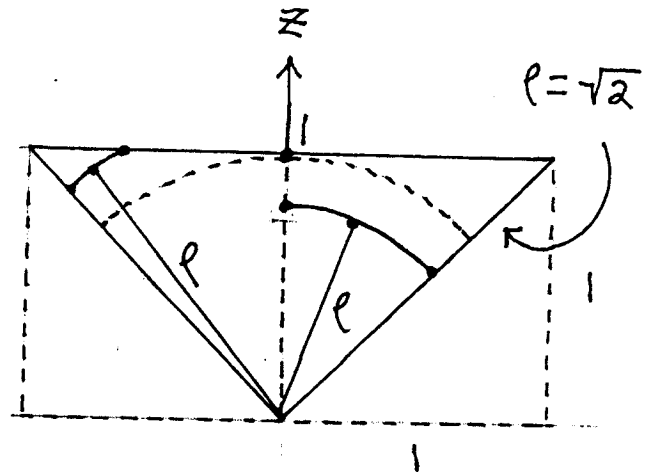
$$z=1 \rightarrow \ell \cos \phi = 1 \rightarrow$$

$$\ell = 1/\cos \phi \rightarrow$$

$$\ell = \sec \phi$$

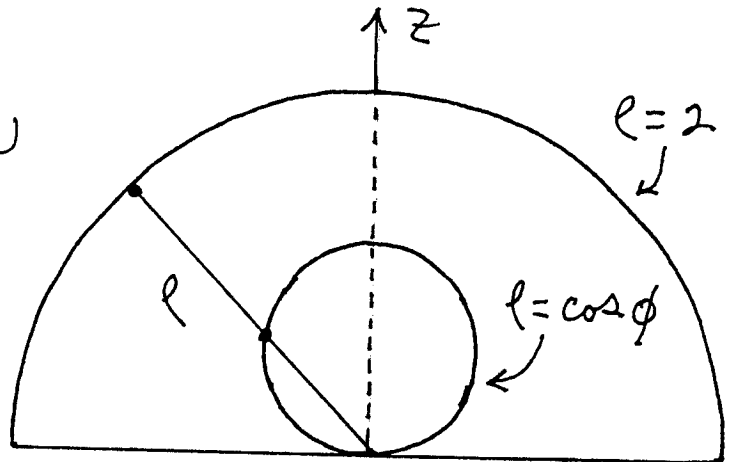
$$a.) Vol = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec \phi} 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

b.) SIDE VIEW  
OF SOLID



$$Vol = \int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\sec \phi}^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

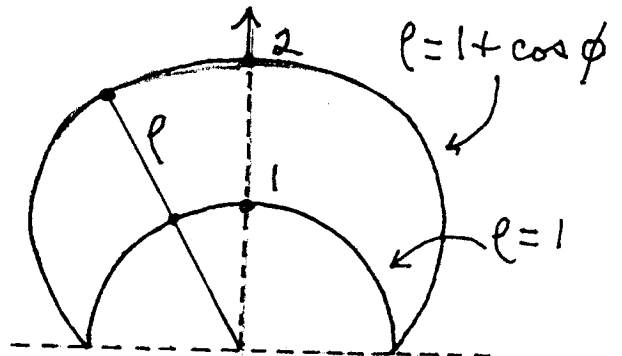
33.) SIDE VIEW  
OF SOLID



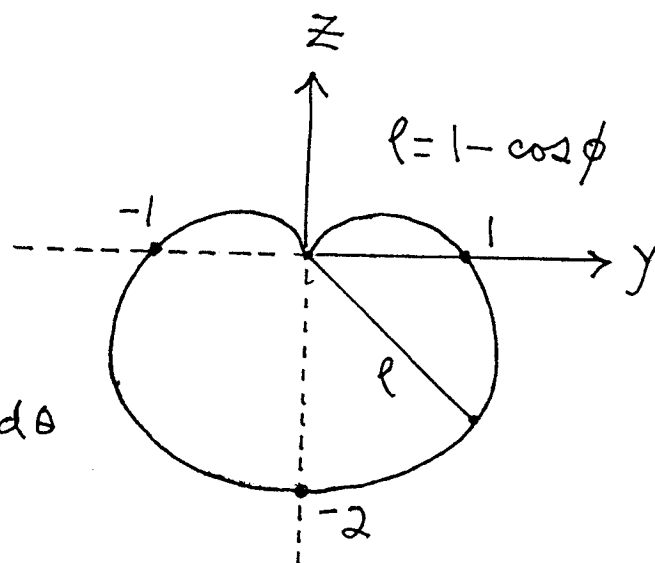
$$Vol = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\cos \phi}^2 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

34.)

$$Vol = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^{1+\cos \phi} 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



35.) SIDE VIEW

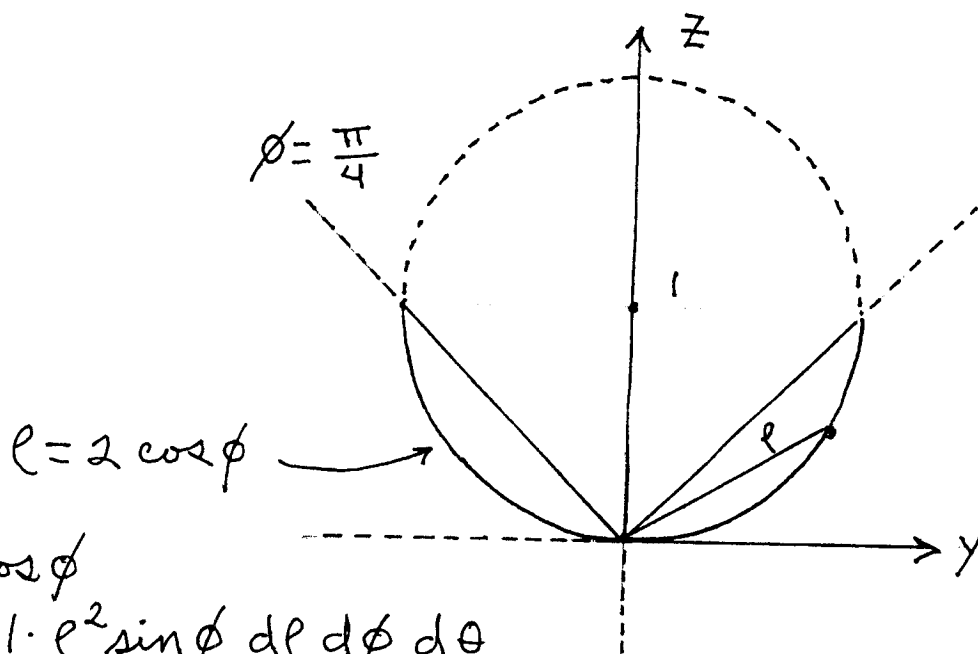


$$Vol = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

37.)  $z = \sqrt{x^2 + y^2}$  (cone)

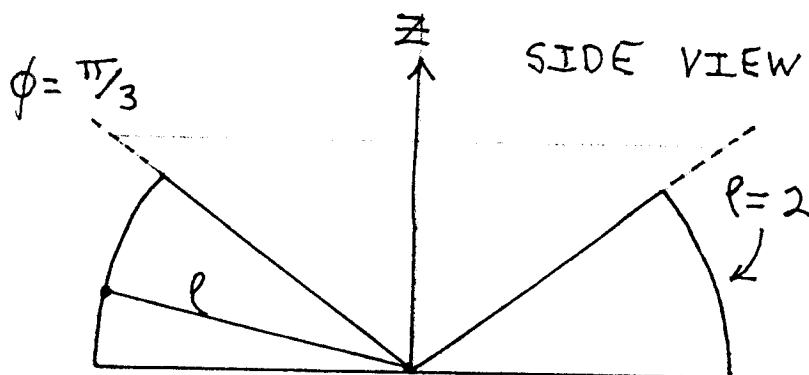
$$\rightarrow \phi = \frac{\pi}{4}$$

SIDE VIEW



$$Vol = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

38.)



$$Vol = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

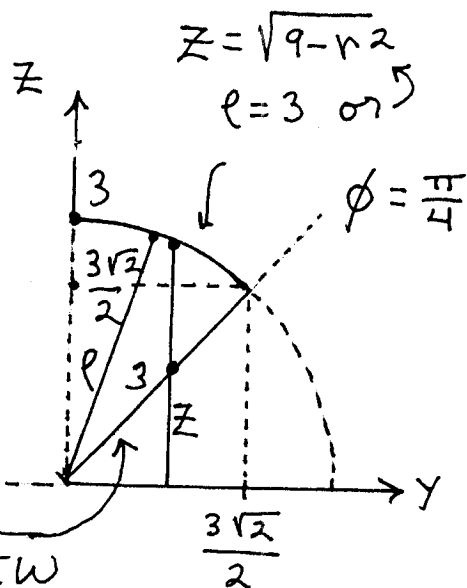
40.)

$$z = \sqrt{x^2 + y^2}$$

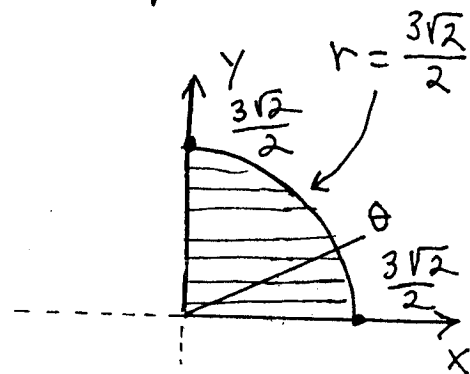
or

$$z = r$$

SIDE VIEW

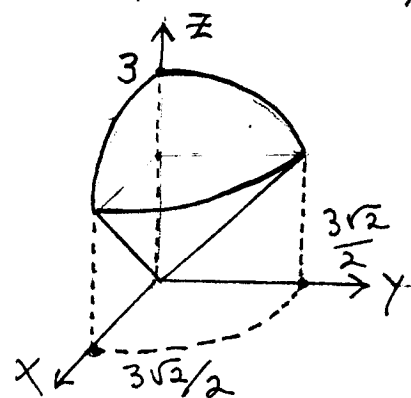


proj. on  
xy-plane

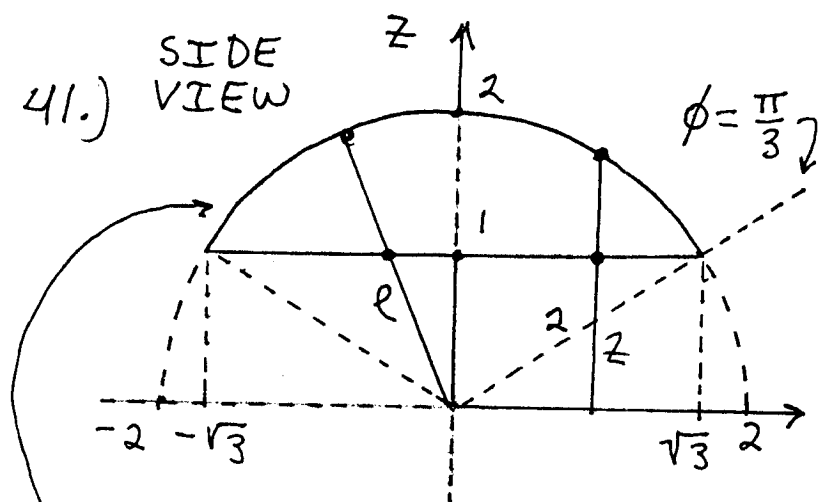


$$a.) Vol = \int_0^{\pi/2} \int_0^{\frac{3\sqrt{2}}{2}} \int_r^{\sqrt{9-r^2}} 1 \cdot r \, dz \, dr \, d\theta$$

$$b.) Vol = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 1 \cdot \ell^2 \sin \phi \, d\ell \, d\phi \, d\theta$$



41.) SIDE VIEW



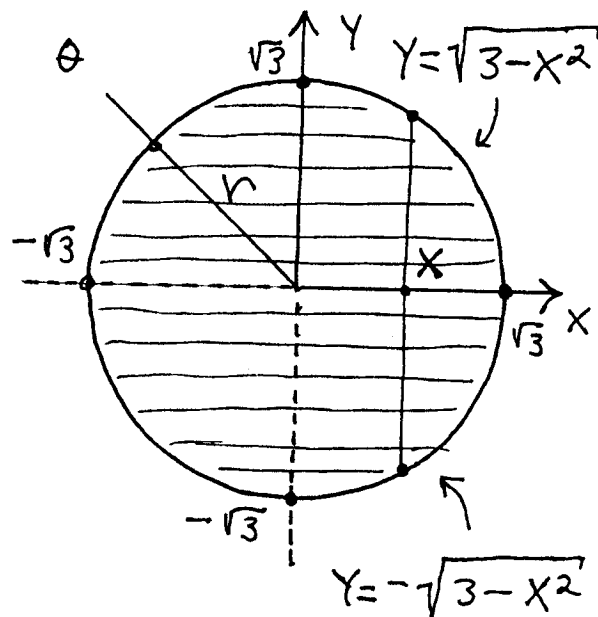
$$z = \sqrt{4 - x^2 - y^2} \text{ or}$$

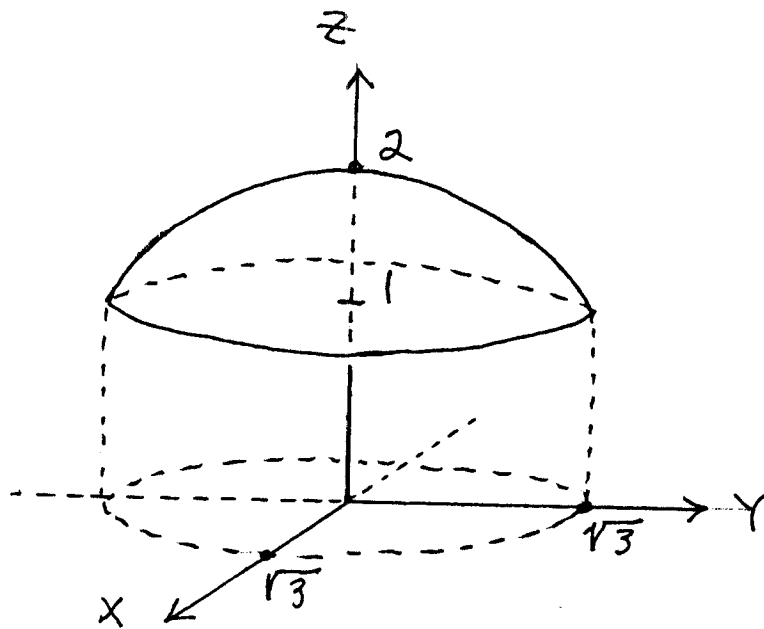
$$z = \sqrt{4 - r^2} \text{ or}$$

$$\ell = 2 ;$$

$$z = 1 \text{ or } \ell \cos \phi = 1 \text{ or } \ell = \sec \phi$$

proj. on xy-plane



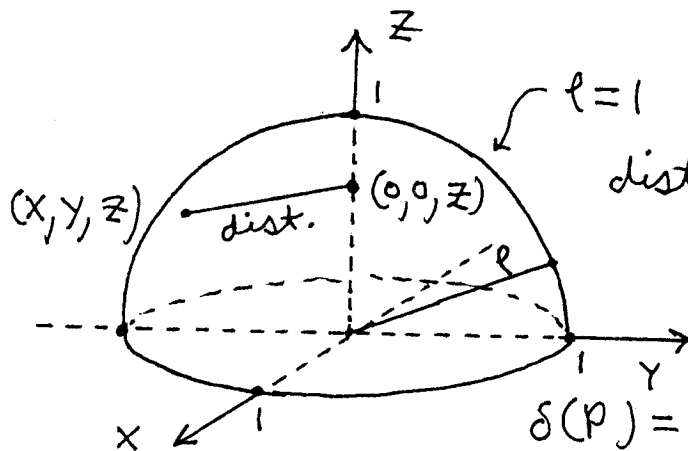


$$a.) Vol = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\sec \phi}^{\ell} 1 \cdot \ell^2 \sin \phi \, d\ell \, d\phi \, d\theta$$

$$b.) Vol = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 1 \cdot r \, dz \, dr \, d\theta$$

$$c.) Vol = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx$$

42.) b.)



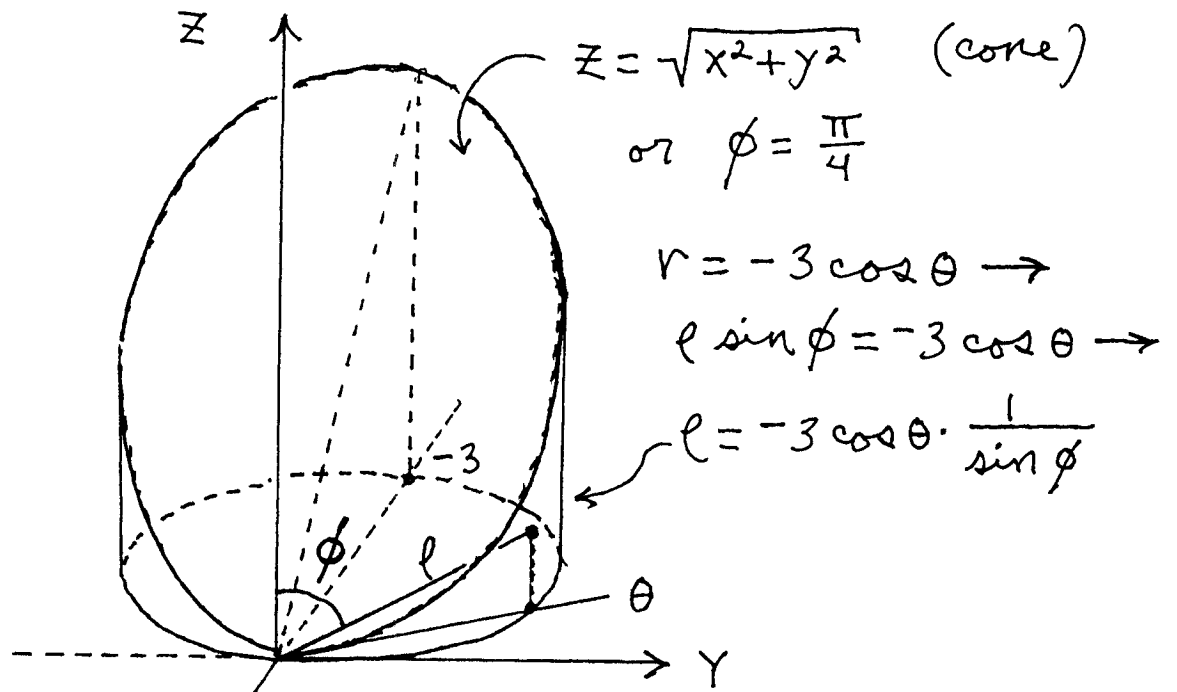
$$\begin{aligned} \text{dist} &= \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2} \\ &= \sqrt{x^2 + y^2} = \sqrt{r^2} \end{aligned}$$

$$= r = \ell \sin \phi,$$

$$\delta(P) = z = \ell \cos \phi;$$

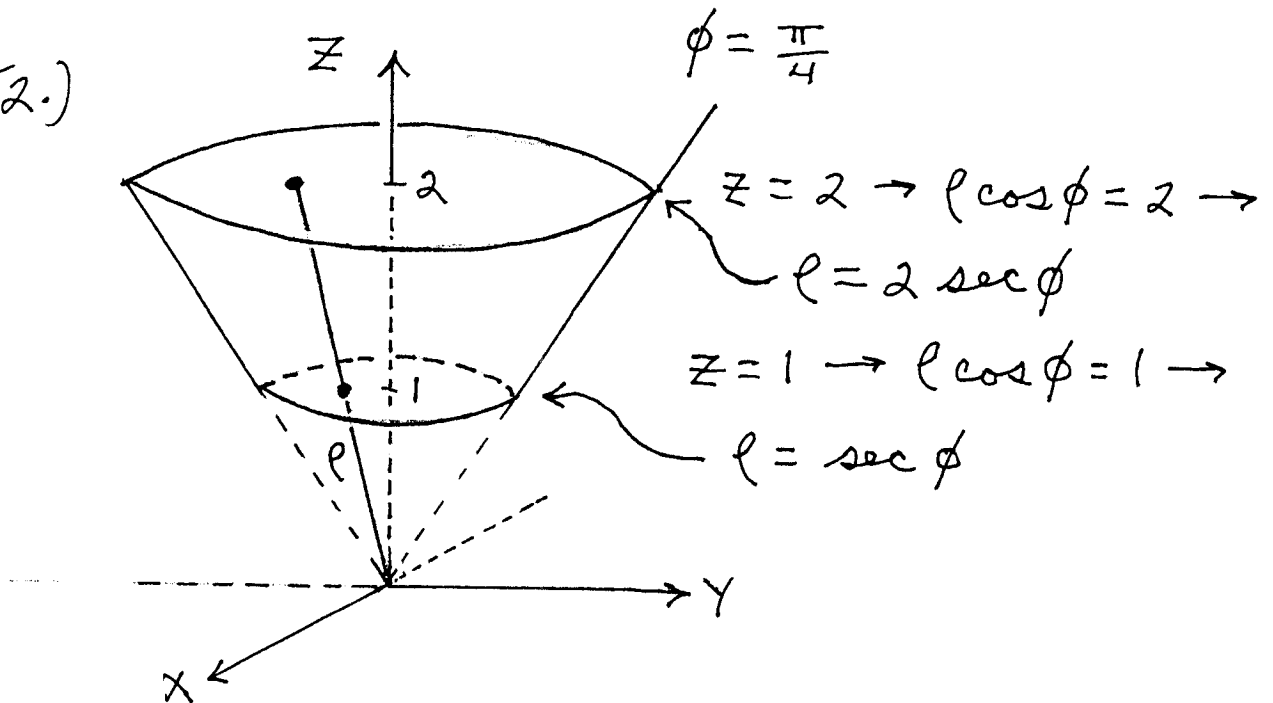
$$I_z = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 (\ell \sin \phi)^2 (\ell \cos \phi) \cdot \ell^2 \sin \phi \cdot d\ell \, d\phi \, d\theta$$

46.)



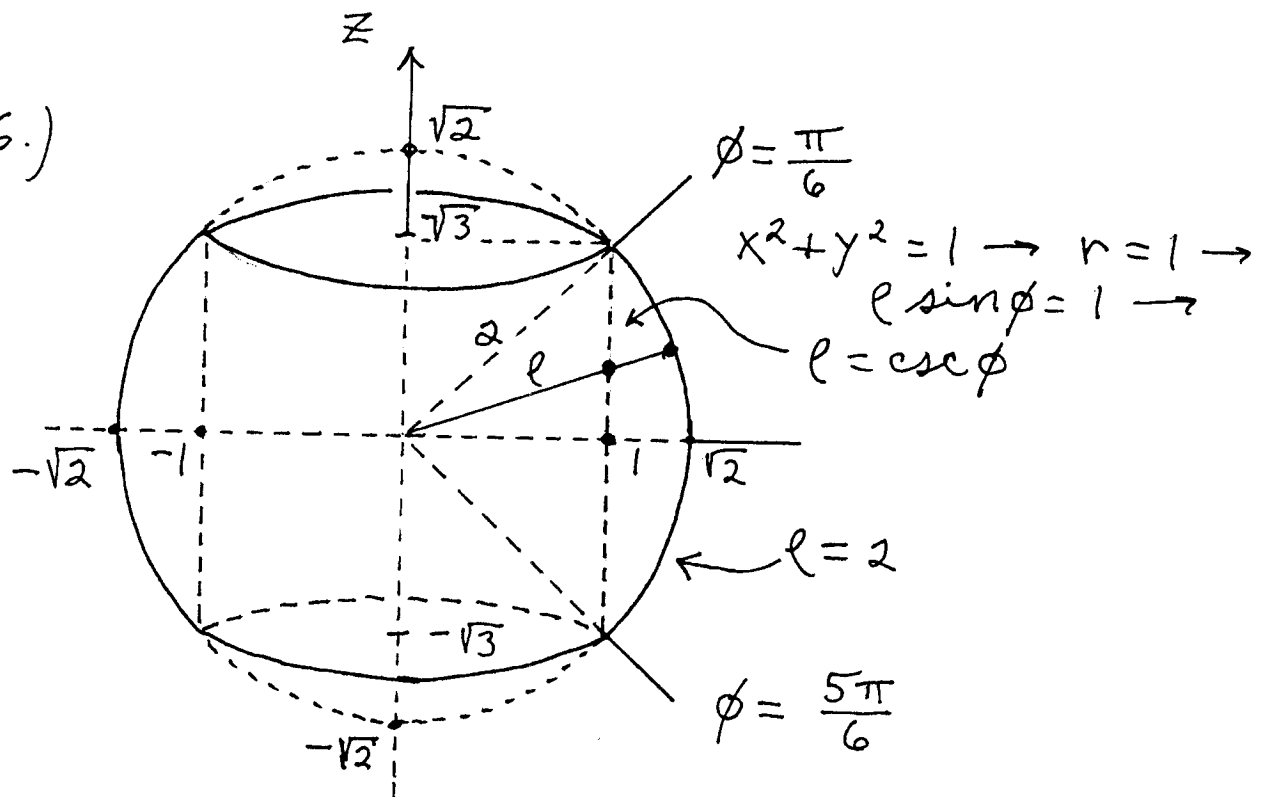
$$\text{Vol} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{-3 \cos \theta \cdot \frac{1}{\sin \phi}} 1 \cdot \rho^2 \sin \phi \cdot d\rho d\phi d\theta$$

52.)



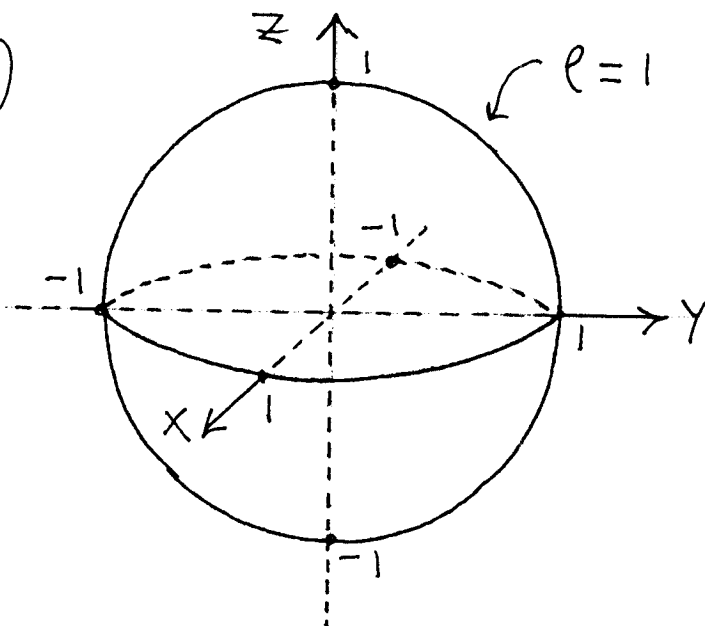
$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\sec \phi}^{2 \sec \phi} 1 \cdot \rho^2 \sin \phi \cdot d\rho d\phi d\theta$$

56.)



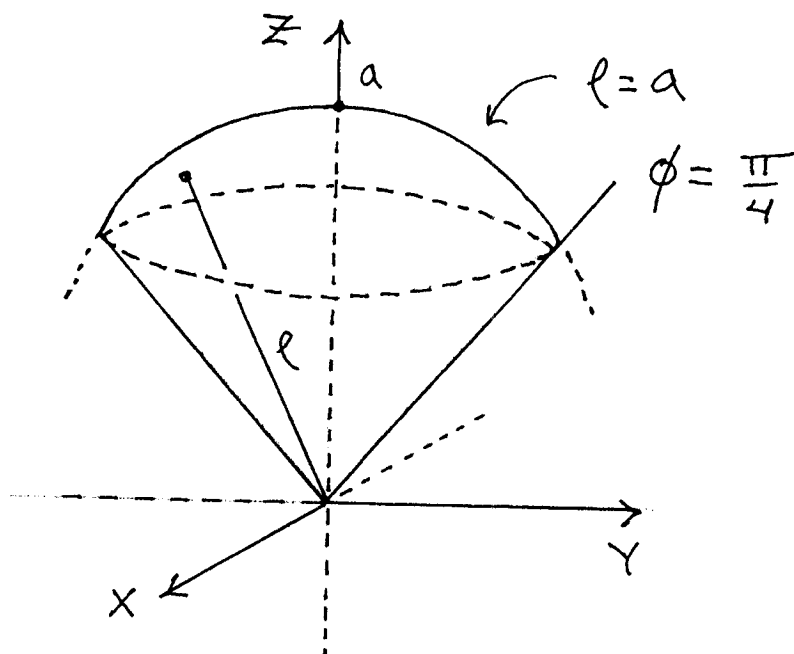
$$\text{Vol} = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc \phi}^2 l \cdot l^2 \sin \phi \cdot dl d\phi d\theta$$

65)



$$\begin{aligned} \text{AVE} &= \frac{1}{\text{vol } R} \iiint_R f(\rho) dV \\ &= \frac{1}{\int_0^{2\pi} \int_0^{\pi} \int_0^1 l \cdot l^2 \sin \phi dl d\phi d\theta} \end{aligned}$$

70.)



$$\bar{X} = \frac{\iiint_R x \, dV}{\iiint_R 1 \, dV}$$

$$= \frac{\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^a (\rho \cdot \sin \phi \cos \theta) \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}{\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^a 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}$$