NOTES ON L^p AND SOBOLEV SPACES

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1. L^p SPACES

1.1. Definitions and basic properties.

Definition 1.1. *Let $0 < p < \infty$ and let $(X, \mathcal{M}, \mu)$ denote a measure space. If $f : X \to \mathbb{R}$ is a measurable function, then we define*

$$\|f\|_{L^p(X)} := \left(\int_X |f|^p dx \right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L^\infty(X)} := \text{ess sup}_{x \in X} |f(x)| .$$

Note that $\|f\|_{L^p(X)}$ may take the value ∞.

Definition 1.2. *The space $L^p(X)$ is the set*

$$L^p(X) = \{ f : X \to \mathbb{R} \mid \|f\|_{L^p(X)} < \infty \} .$$

The space $L^p(X)$ satisfies the following vector space properties:

1. For each $\alpha \in \mathbb{R}$, if $f \in L^p(X)$ then $\alpha f \in L^p(X)$;
2. If $f, g \in L^p(X)$, then
$$|f + g|^p \le 2^{p-1} (|f|^p + |g|^p) ,$$
so that $f + g \in L^p(X)$.
3. The triangle inequality is valid if $p \ge 1$.

The most interesting cases are $p = 1, 2, \infty$, while all of the L^p arise often in *nonlinear* estimates.

Definition 1.3. *The space l^p, called "little L^p", will be useful when we introduce Sobolev spaces on the torus and the Fourier series. For $1 \le p < \infty$, we set*

$$l^p = \left\{ \{x_n\}_{n=1}^\infty \mid \sum_{n=1}^\infty |x_n|^p < \infty \right\} .$$

1.2. Basic inequalities.

Lemma 1.4. *For $\lambda \in (0, 1)$, $x^\lambda \le (1 - \lambda) + \lambda x$.*

Proof. Set $f(x) = (1 - \lambda) + \lambda x - x^\lambda$; hence, $f'(x) = \lambda - \lambda x^{\lambda - 1} = 0$ if and only if $\lambda(1 - x^{\lambda-1}) = 0$ so that $x = 1$ is the critical point of f. In particular, the minimum occurs at $x = 1$ with value

$$f(1) = 0 \le (1 - \lambda) + \lambda x - x^\lambda .$$

□

Lemma 1.5. *For $a, b \ge 0$ and $\lambda \in (0, 1)$, $a^\lambda b^{1-\lambda} \le \lambda a + (1 - \lambda) b$ with equality if $a = b$.*

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Proof. If either $a = 0$ or $b = 0$, then this is trivially true, so assume that $a, b > 0$. Set $x = a/b$, and apply Lemma 1 to obtain the desired inequality. $\square$

Theorem 1.6 (Hölder's inequality). *Suppose that $1 \leq p \leq \infty$ and $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$. Moreover,*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} .$$

Note that if $p = q = 2$, then this is the Cauchy-Schwarz inequality since $\|fg\|_{L^1} = |(f, g)_{L^2}|$.

Proof. We use Lemma 1.5. Let $\lambda = 1/p$ and set

$$a = \frac{|f|^p}{\|f\|^p_{L^p}} , \text{ and } \quad b = \frac{|g|^q}{\|g\|^q_{L^p}}$$

for all $x \in X$. Then $a^\lambda b^{1-\lambda} = a^{1/p} b^{1-1/p} = a^{1/p} b^{1/q}$ so that

$$\frac{|f| \cdot |g|}{\|f\|_{L^p} \|g\|_{L^q}} \leq \frac{1}{p} \frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q} \frac{|g|^q}{\|g\|^q_{L^q}} .$$

Integrating this inequality yields

$$\int_X \frac{|f| \cdot |g|}{\|f\|_{L^p} \|g\|_{L^q}} dx \leq \int_X \left(\frac{1}{p} \frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q} \frac{|g|^q}{\|g\|^q_{L^q}} \right) dx = \frac{1}{p} + \frac{1}{q} = 1 .$$

$\square$

Definition 1.7. *$q = \frac{p}{p-1}$ or $\frac{1}{q} = 1 - \frac{1}{p}$ is called the conjugate exponent of p.*

Theorem 1.8 (Minkowski's inequality). *If $1 \leq p \leq \infty$ and $f, g \in L^p$ then*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p} .$$

Proof. If $f + g = 0$ a.e., then the statement is trivial. Assume that $f + g \neq 0$ a.e. Consider the equality

$$|f + g|^p = |f + g| \cdot |f + g|^{p-1} \leq (|f| + |g|)|f + g|^{p-1} ,$$

and integrate over X to find that

$$\int_X |f + g|^p dx \leq \int_X \left[(|f| + |g|)|f + g|^{p-1} \right] dx$$
$$\overset{\text{Hölder's}}{\leq} (\|f\|_{L^p} + \|g\|_{L^p}) \left\| |f + g|^{p-1} \right\|_{L^q} .$$

Since $q = \frac{p}{p-1}$,

$$\left\| |f + g|^{p-1} \right\|_{L^q} = \left(\int_X |f + g|^p dx \right)^{\frac{1}{q}} ,$$

from which it follows that

$$\left(\int_X |f + g|^p dx \right)^{1 - \frac{1}{q}} \leq \|f\|_{L^p} + \|g\|_{L^q} ,$$

which completes the proof, since $\frac{1}{p} = 1 - \frac{1}{q}$. $\square$

Corollary 1.9. *For $1 \leq p \leq \infty$, $L^p(X)$ is a normed linear space.*

Example 1.10. *Let Ω denote a (Lebesgue) measure-1 subset of $\mathbb{R}^n$. If $f \in L^1(\Omega)$ satisfies $f(x) \geq M > 0$ for almost all $x \in \Omega$, then $\log(f) \in L^1(\Omega)$ and satisfies*

$$\int_\Omega \log f dx \leq \log(\int_\Omega f dx) .$$

To see this, consider the function $g(t) = t - 1 - \log t$ for $t > 0$. Compute $g'(t) = 1 - \frac{1}{t} = 0$ so $t = 1$ is a minimum (since $g''(1) > 0$). Thus, $\log t \leq t - 1$ and letting $t \mapsto \frac{1}{t}$ we see that

$$1 - \frac{1}{t} \leq \log t \leq t - 1 . \tag{1.1}$$

Since $\log x$ is continuous and f is measurable, then $\log f$ is measurable for $f > 0$. Let $t = \frac{f(x)}{\|f\|_{L^1}}$ in (1.1) to find that

$$1 - \frac{\|f\|_{L^1}}{f(x)} \leq \log f(x) - \log \|f\|_{L^1} \leq \frac{f(x)}{\|f\|_{L^1}} - 1 . \tag{1.2}$$

Since $g(x) \leq \log f(x) \leq h(x)$ for two integrable functions g and h, it follows that $\log f(x)$ is integrable. Next, integrate (1.2) to finish the proof, as $\int_X \left(\frac{f(x)}{\|f\|_{L^1}} - 1 \right) dx = 0$.

1.3. **The space** $(L^p(X), \|\cdot\|_{L^p}(X)$ **is complete.** Recall the a normed linear space is a Banach space if every Cauchy sequence has a limit in that space; furthermore, recall that a sequence $x_n \to x$ in X if $\lim_{n\to\infty} \|x_n - x\|_X = 0$.

The proof of completeness makes use of the following two lemmas which are restatements of the Monotone Convergence Theorem and the Dominated Convergence Theorem, respectively.

Lemma 1.11 (MCT). *If $f_n \in L^1(X)$, $0 \leq f_1(x) \leq f_2(x) \leq \cdots$, and $\|f_n\|_{L^1(X)} \leq C < \infty$, then $\lim_{n\to\infty} f_n(x) = f(x)$ with $f \in L^1(X)$ and $\|f_n - f\|_{L^1} \to 0$ as $n \to 0$.*

Lemma 1.12 (DCT). *If $f_n \in L^1(X)$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., and if $\exists\ g \in L^1(X)$ such that $|f_n(x)| \leq |g(x)|$ a.e. for all n, then $f \in L^1(X)$ and $\|f_n - f\|_{L^1} \to 0$.*

Proof. Apply the Dominated Convergene Theorem to the sequence $h_n = |f_n - f| \to 0$ a.e., and note that $|h_n| \leq 2g$. ☐

Theorem 1.13. *If $1 \leq p < \infty$ then $L^p(X)$ is a Banach space.*

Proof. **Step 1. The Cauchy sequence.** Let $\{f_n\}_{n=1}^\infty$ denote a Cauchy sequence in L^p, and assume without loss of generality (by extracting a subsequence if necessary) that $\|f_{n+1} - f_n\|_{L^p} \leq 2^{-n}$.

Step 2. Conversion to a convergent monotone sequence. Define the sequence $\{g_n\}_{n=1}^\infty$ as

$$g_1 = 0, \quad g_n = |f_1| + |f_2 - f_1| + \cdots + |f_n - f_{n-1}| \quad \text{for} \quad n \geq 2 .$$

It follows that

$$0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \cdots$$

so that g_n is a monotonically increasing sequence. Furthermore, $\{g_n\}$ is uniformly bounded in L^p as

$$\int_X g_n^p dx = \|g_n\|_{L^p}^p \leq \left(\|f_1\|_{L^p} + \sum_{i=2}^\infty \|f_i - f_{i-1}\|_{L^p} \right)^p \leq (\|f\|_{L^p} + 1)^p ;$$

thus, by the Monotone Convergence Theorem, $g_n^p \nearrow g^p$ a.e., $g \in L^p$, and $g_n \leq g$ a.e.

Step 3. Pointwise convergence of $\{f_n\}$. For all $k \geq 1$,

$$\begin{aligned} |f_{n+k} - f_n| &= |f_{n+k} - f_{n+k-1} + f_{n+k-1} + \cdots - f_{n+1} + f_{n+1} - f_n| \\ &\leq \sum_{i=n+1}^{k+1} |f_i - f_{i-1}| = g_{n+k} - g_n \longrightarrow 0 \text{ a.e.} \end{aligned}$$

Therefore, $f_n \to f$ a.e. Since

$$|f_n| \leq |f_1| + \sum_{i=2}^{n} |f_i - f_{i-1}| \leq g_n \leq g \text{ for all } n \in \mathbb{N},$$

it follows that $|f| \leq g$ a.e. Hence, $|f_n|^p \leq g^p$, $|f|^p \leq g^p$, and $|f - f_n|^p \leq 2g^p$, and by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_X |f - f_n|^p dx = \int_X \lim_{n\to\infty} |f - f_n|^p dx = 0\,.$$

□

1.4. **Convergence criteria for L^p functions.** If $\{f_n\}$ is a sequence in $L^p(X)$ which converges to f in $L^p(X)$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ for almost every $x \in X$ (denoted by a.e.), but it is in general *not true* that the entire sequence itself will converge pointwise a.e. to the limit f, without some further conditions holding.

Example 1.14. *Let $X = [0, 1]$, and consider the subintervals*

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right], \left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{2}{4}\right], \left[\frac{2}{4}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], \left[0, \frac{1}{5}\right], \cdots$$

Let f_n denote the indicator function of the n^{th} interval of the above sequence. Then $\|f_n\|_{L^p} \to 0$, but $f_n(x)$ does not converge for any $x \in [0, 1]$.

Example 1.15. *Set $X = \mathbb{R}$, and for $n \in \mathbb{N}$, set $f_n = \mathbf{1}_{[n,n+1]}$. Then $f_n(x) \to 0$ as $n \to \infty$, but $\|f_n\|_{L^p} = 1$ for $p \in [1, \infty)$; thus, $f_n \to 0$ pointwise, but not in L^p.*

Example 1.16. *Set $X = [0, 1]$, and for $n \in \mathbb{N}$, set $f_n = n\mathbf{1}_{[0,\frac{1}{n}]}$. Then $f_n(x) \to 0$ a.e. as $n \to \infty$, but $\|f_n\|_{L^1} = 1$; thus, $f_n \to 0$ pointwise, but not in L^p.*

Theorem 1.17. *For $1 \leq p < \infty$, suppose that $\{f_n\} \subset L^p(X)$ and that $f_n(x) \to f(x)$ a.e. If $\lim_{n\to\infty} \|f_n\|_{L^p(X)} = \|f\|_{L^p(X)}$, then $f_n \to f$ in $L^p(X)$.*

Proof. Given $a, b \geq 0$, convexity implies that $\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}(a^p + b^p)$ so that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, and hence $|a-b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. Set $a = f_n$ and $b = f$ to obtain the inequality

$$0 \leq 2^{p-1}\left(|f_n|^p + |f|^p\right) - |f_n - f|^p\,.$$

Since $f_n(x) \to f(x)$ a.e.,

$$2^p \int_X |f|^p dx = \int_X \lim_{n\to\infty} \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right) dx\,.$$

Thus, Fatou's lemma asserts that

$$2^p \int_X |f|^p dx \leq \liminf_{n\to\infty} \int_X \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right) dx$$

Since $\|f_n\|_{L^p(X)} \to \|f\|_{L^p(X)}$, we see $\limsup_{n\to\infty} \|f_n - f\|_{L^p(X)} = 0$. $\square$

1.5. The space $L^\infty(X)$.

Definition 1.18. *With* $\|f\|_{L^\infty(X)} = \inf\{M \geq 0 \mid |f(x)| \leq M \quad a.e.\}$, *we set*

$$L^\infty(X) = \{f : X \to \mathbb{R} \mid \|f\|_{L^\infty(X)} < \infty\}\,.$$

Theorem 1.19. $(L^\infty(X), \|\cdot\|_{L^\infty(X)})$ *is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $L^\infty(X)$. It follows that $|f_n - f_m| \leq \|f_n - f_m\|_{L^\infty(X)}$ a.e. and hence $f_n(x) \to f(x)$ a.e., where f is measurable and essentially bounded.

Choose $\epsilon > 0$ and $N(\epsilon)$ such that $\|f_n - f_m\|_{L^\infty(X)} < \epsilon$ for all $n, m \geq N(\epsilon)$. Since $|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \epsilon$ holds a.e. $x \in X$, it follows that $\|f - f_n\|_{L^\infty(X)} \leq \epsilon$ for $n \geq N(\epsilon)$, so that $\|f_n - f\|_{L^\infty(X)} \to 0$. $\square$

Remark 1.20. *In general, there is no relation of the type* $L^p \subset L^q$. *For example, suppose that* $X = (0,1)$ *and set* $f(x) = x^{-\frac{1}{2}}$. *Then* $f \in L^1(0,1)$, *but* $f \notin L^2(0,1)$. *On the other hand, if* $X = (1,\infty)$ *and* $f(x) = x^{-1}$, *then* $f \in L^2(1,\infty)$, *but* $f \notin L^1(1,\infty)$.

Lemma 1.21 (L^p comparisons)**.** *If* $1 \leq p < q < r \leq \infty$, *then (a)* $L^p \cap L^r \subset L^q$, *and (b)* $L^q \subset L^p + L^r$.

Proof. We begin with (b). Suppose that $f \in L^q$, define the set $E = \{x \in X \ : |f(x)| \geq 1\}$, and write f as

$$\begin{aligned} f &= f\mathbf{1}_E + f\mathbf{1}_{E^c} \\ &= g + h\,. \end{aligned}$$

Our goal is to show that $g \in L^p$ and $h \in L^r$. Since $|g|^p = |f|^p\mathbf{1}_E \leq |f|^q\mathbf{1}_E$ and $|h|^r = |f|^r\mathbf{1}_{E^c} \leq |f|^q\mathbf{1}_{E^c}$, assertion (b) is proven.

For (a), let $\lambda \in [0,1]$ and for $f \in L^q$,

$$\begin{aligned} \|f\|_{L^q} = \left(\int_X |f|^q dx\right)^{\frac{1}{q}} &= \left(\int_X |f|^{\lambda q}|f|^{(1-\lambda)q} d\mu\right)^{\frac{1}{q}} \\ &\leq \left(\|f\|^{\lambda q}_{L^p}\|f\|^{(1-\lambda)q}_{L^r}\right)^{\frac{1}{q}} = \|f\|^{\lambda}_{L^p}\|f\|^{(1-\lambda)}_{L^r}\,. \end{aligned}$$

$\square$

Theorem 1.22. *If* $\mu(X) \leq \infty$ *and* $q > p$, *then* $L^q \subset L^p$.

Proof. Consider the case that $q = 2$ and $p = 1$. Then by the Cauchy-Schwarz inequality,

$$\int_X |f| dx = \int_X |f| \cdot 1\, dx \leq \|f\|_{L^2(X)} \sqrt{\mu(X)}\,.$$

$\square$

1.6. **Approximation of $L^p(X)$ by simple functions.**

Lemma 1.23. *If $p \in [1,\infty)$, then the set of simple functions $f = \sum_{i=1}^n a_i \mathbf{1}_{E_i}$, where each E_i is an element of the σ-algebra $\mathcal{A}$ and $\mu(E_i) < \infty$, is dense in $L^p(X, \mathcal{A}, \mu)$.*

Proof. If $f \in L^p$, then f is measurable; thus, there exists a sequence $\{\phi_n\}_{n=1}^\infty$ of simple functions, such that $\phi_n \to f$ a.e. with

$$0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|,$$

i.e., ϕ_n approximates f from below.

Recall that $|\phi_n - f|^p \to 0$ a.e. and $|\phi_n - f|^p \le 2^p |f|^p \in L^1$, so by the Dominated Convergence Theorem, $\|\phi_n - f\|_{L^p} \to 0$.

Now, suppose that the set E_i are disjoint; then b definition of the Lebesgue integral,

$$\int_X \phi_n^p dx = \sum_{i=1}^n |a_i|^p \mu(E_i) < \infty\,.$$

If $a_i \ne 0$, then $\mu(E_i) < \infty$. □

1.7. **Approximation of $L^p(\Omega)$ by continuous functions.**

Lemma 1.24. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded. Then $C^0(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1,\infty)$.*

Proof. Let K be any compact subset of Ω. The functions

$$F_{K,n}(x) = \frac{1}{1 + n \operatorname{dist}(x,K)} \in C^0(\Omega) \ \text{ satisfy } \ F_{K,n} \le 1\,,$$

and decrease monotonically to the characteristic function $\mathbf{1}_K$. The Monotone Convergence Theorem gives

$$f_{K,n} \to \mathbf{1}_K \text{ in } L^p(\Omega), \quad 1 \le p < \infty\,.$$

Next, let $A \subset \Omega$ be any measurable set, and let λ denote the Lebesgue measure. Then

$$\lambda(A) = \sup\{\mu(K) \ : \ K \subset A, \quad K \text{ compact}\}\,.$$

It follows that there exists an increasing sequence of K_j of compact subsets of A such that $\lambda(A \backslash \cup_j K_j) = 0$. By the Monotone Convergence Theorem, $\mathbf{1}_{K_j} \to \mathbf{1}_A$ in $L^p(\Omega)$ for $p \in [1,\infty)$. According to Lemma 1.23, each function in $L^p(\Omega)$ is a norm limit of simple functions, so the lemma is proved. □

1.8. **Approximation of $L^p(\Omega)$ by smooth functions.** For $\Omega \subset \mathbb{R}^n$ open, for $\epsilon > 0$ taken sufficiently small, define the open subset of Ω by

$$\Omega_\epsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}\,.$$

Definition 1.25 (Mollifiers)**.** *Define $\eta \in C^\infty(\mathbb{R}^n)$ by*

$$\eta(x) := \begin{cases} Ce^{(|x|^2-1)^{-1}} & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \ge 1 \end{cases}\,,$$

with constant $C > 0$ chosen such that $\int_{\mathbb{R}^n} \eta(x) dx = 1$.

For $\epsilon > 0$, the standard sequence of mollifiers on $\mathbb{R}^n$ is defined by

$$\eta_\epsilon(x) = \epsilon^{-n} \eta(x/\epsilon)\,,$$

and satisfy $\int_{\mathbb{R}^n} \eta_\epsilon(x) dx = 1$ and $\operatorname{spt}(\eta_\epsilon) \subset \overline{B(0,\epsilon)}$.

Definition 1.26. *For $\Omega \subset \mathbb{R}^n$ open, set*

$$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^p(\tilde{\Omega}) \ \ \forall\, \tilde{\Omega} \subset\subset \Omega\}\,,$$

where $\tilde{\Omega} \subset\subset \Omega$ means that there exists K compact such that $\tilde{\Omega} \subset K \subset \Omega$. We say that $\tilde{\Omega}$ is compactly contained in Ω.

Definition 1.27 (Mollification of L^1)**.** *If $f \in L^1_{\text{loc}}(\Omega)$, define its mollification*

$$f^\epsilon = \eta_\epsilon * f \ \text{ in } \Omega_\epsilon\,,$$

so that

$$f^\epsilon(x) = \int_\Omega \eta_\epsilon(x-y) f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y)dy \quad \forall x \in \Omega_\epsilon\,.$$

Theorem 1.28 (Mollification of $L^p(\Omega)$)**.**

(A) $f^\epsilon \in C^\infty(\Omega_\epsilon)$.
(B) $f^\epsilon \to f$ *a.e. as* $\epsilon \to 0$.
(C) *If $f \in C^0(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.*
(D) *If $p \in [1,\infty)$ and $f \in L^p_{\text{loc}}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$.*

Proof. **Part (A).** We rely on the difference quotient approximation of the partial derivative. Fix $x \in \Omega_\epsilon$, and choose h sufficiently small so that $x + he_i \in \Omega_\epsilon$ for $i = 1, ..., n$, and compute the difference quotient of f^ϵ:

$$\begin{aligned} \frac{f^\epsilon(x+he_i) - f(x)}{\epsilon} &= \epsilon^{-n} \int_\Omega \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{h}\right)\right] f(y)dy \\ &= \epsilon^{-n} \int_{\tilde{\Omega}} \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)dy \end{aligned}$$

for some open set $\tilde{\Omega} \subset\subset \Omega$. On $\tilde{\Omega}$,

$$\lim_{h\to 0} \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] = \frac{1}{\epsilon}\frac{\partial\eta}{\partial x_i}\left(\frac{x-y}{\epsilon}\right) = \epsilon^n \frac{\partial\eta_\epsilon}{\partial x_i}\left(\frac{x-y}{\epsilon}\right)\,,$$

so by the Dominated Convergence Theorem,

$$\frac{\partial f_\epsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial \eta_\epsilon}{\partial x_i}(x-y) f(y)dy\,.$$

A similar argument for higher-order partial derivatives proves (A).

Step 2. Part (B). By the Lebesgue differentiation theorem,

$$\lim_{\epsilon\to 0} \frac{1}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} |f(y) - f(x)|dy \text{ for a.e. } x \in \Omega\,.$$

Choose $x \in \Omega$ for which this limit holds. Then

$$\begin{aligned} |f_\epsilon(x) - f(x)| &\le \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)-f(x)|dy \\ &= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta((x-y)/\epsilon)|f(y)-f(x)|dy \\ &\le \frac{C}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} |f(x)-f(y)|dy \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0\,. \end{aligned}$$

Step 3. Part (C). For $\tilde{\Omega} \subset \Omega$, the above inequality shows that if $f \in C^0(\Omega)$ and hence uniformly continuous on $\tilde{\Omega}$, then $f^\epsilon(x) \to f(x)$ uniformly on $\tilde{\Omega}$.

Step 4. Part (D). For $f \in L^p_{\text{loc}}(\Omega)$, $p \in [1, \infty)$, choose open sets $U \subset\subset D \subset\subset \Omega$; then, for $\epsilon > 0$ small enough,

$$\|f^\epsilon\|_{L^p(U)} \leq \|f\|_{L^p(D)} .$$

To see this, note that

$$\begin{aligned} |f^\epsilon(x)| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|dy \\ &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{(p-1)/p}\eta_\epsilon(x-y)^{1/p}|f(y)|dy \\ &\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy\right)^{(p-1)/p} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy\right)^{1/p} , \end{aligned}$$

so that for $\epsilon > 0$ sufficiently small

$$\begin{aligned} \int_U |f^\epsilon(x)|^p dx &\leq \int_U \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy dx \\ &\leq \int_D |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)dx\right) dy \leq \int_D |f(y)|^p dy . \end{aligned}$$

Since $C^0(D)$ is dense in $L^p(D)$, choose $g \in C^0(D)$ such that $\|f-g\|_{L^p(D)} < \delta$; thus

$$\begin{aligned} \|f^\epsilon - f\|_{L^p(U)} &\leq \|f^\epsilon - g^\epsilon\|_{L^p(U)} + \|g^\epsilon - g\|_{L^p(U)} + \|g-f\|_{L^p(U)} \\ &\leq 2\|f-g\|_{L^p(D)} + \|g^\epsilon - g\|_{L^p(U)} \leq 2\delta + \|g^\epsilon - g\|_{L^p(U)} . \end{aligned}$$

□

1.9. **Continuous linear functionals on $L^p(X)$.** Let $L^p(X)'$ denote the dual space of $L^p(X)$. For $\phi \in L^p(X)'$, the *operator norm* of ϕ is defined by $\|\phi\|_{\text{op}} = \sup_{L^p(X)=1} |\phi(f)|$.

Theorem 1.29. *Let $p \in (1, \infty]$, $q = \frac{p}{p-1}$. For $g \in L^q(X)$, define $F_g : L^p(X) \to \mathbb{R}$ as*

$$F_g(f) = \int_X fg dx .$$

Then F_g is a continuous linear functional on $L^p(X)$ with operator norm $\|F_g\|_{\text{op}} = \|g\|_{L^p(X)}$.

Proof. The linearity of F_g again follows from the linearity of the Lebesgue integral. Since

$$|F_g(f)| = \left|\int_X fg dx\right| \leq \int_X |fg|\, dx \leq \|f\|_{L^p} \|g\|_{L^q} ,$$

with the last inequality following from Hölder's inequality, we have that $\sup_{\|f\|_{L^p}=1} |F_g(f)| \leq \|g\|_{L^q}$.

For the reverse inequality let $f = |g|^{q-1} \operatorname{sgn} g$. f is measurable and in L^p since $|f|^p = |f|^{\frac{q}{q-1}} = |g|^q$ and since $fg = |g|^q$,

$$F_g(f) = \int_X fg dx = \int_X |g|^q dx = \left(\int_X |g|^q \, dx\right)^{\frac{1}{p}+\frac{1}{q}}$$
$$= \left(\int_X |f|^p \, dx\right)^{\frac{1}{p}} \left(\int_X |g^q| \, dx\right)^{\frac{1}{q}} = \|f\|_{L^p} \|g\|_{L^q}$$

so that $\|g\|_{L^q} = \dfrac{F_g(f)}{\|f\|_{L^p}} \leq \|F_g\|_{\mathrm{op}}$.

□

Remark 1.30. *Theorem 1.29 shows that for $1 < p \leq \infty$, there exists a linear isometry $g \mapsto F_g$ from $L^q(X)$ into $L^p(X)'$, the dual space of $L^p(X)$. When $p = \infty$, $g \mapsto F_g : L^1(X) \to L^\infty(X)'$ is rarely onto ($L^\infty(X)'$ is strictly larger than $L^1(X)$); on the other hand, if the measure space X is σ-finite, then $L^\infty(X) = L^1(X)'$.*

1.10. **A theorem of F. Riesz.**

Theorem 1.31 (Representation theorem)**.** *Suppose that $1 < p < \infty$ and $\phi \in L^p(X)'$. Then there exists $g \in L^q(X)$, $q = \frac{p}{p-1}$ such that*

$$\phi(f) = \int_X fg dx \quad \forall f \in L^p(X),$$

and $\|\phi\|_{\mathrm{op}} = \|g\|_{L^q}$.

Corollary 1.32. *For $p \in (1, \infty)$ the space $L^p(X, \mu)$ is reflexive, i.e., $L^p(X)'' = L^p(X)$.*

The proof Theorem 1.31 crucially relies on the Radon-Nikodym theorem, whose statement requires the following definition.

Definition 1.33. *If μ and ν are measure on (X, A) then $\nu \ll \mu$ if $\nu(E) = 0$ for every set E for which $\mu(E) = 0$. In this case, we say that ν is absolutely continuous with respect to μ.*

Theorem 1.34 (Radon-Nikodym)**.** *If μ and ν are two finite measures on X, i.e., $\mu(X) < \infty, \nu(X) < \infty$, and $\nu \ll \mu$, then*

$$\int_X F(x) \, d\nu(x) = \int_X F(x) h(x) d\mu(x) \tag{1.3}$$

holds for some nonnegative function $h \in L^1(X, \mu)$ and every positive measurable function F.

Proof. Define measures $\alpha = \mu + 2\nu$ and $\omega = 2\mu + \nu$, and let $\mathcal{H} = L^2(X, \alpha)$ (a Hilbert space) and suppose $\phi : L^2(X, \alpha) \to \mathbb{R}$ is defined by $\phi(f) = \int_X f d\omega$. We show that ϕ is a bounded linear functional since

$$|\phi(f)| = \left|\int_X f \, d(2\mu + \nu)\right| \leq \int_X |f| \, d(2\mu + 4\nu) = 2 \int_X |f| \, d\alpha$$
$$\leq \|f\|_{L^2(x,\alpha)} \sqrt{\alpha(X)}.$$

Thus, by the Riesz representation theorem, there exists $g \in L^2(X,\alpha)$ such that

$$\phi(f) = \int_X f\, d\omega = \int_X fg\, d\alpha\,,$$

which implies that

$$\int_X f(2g-1)d\nu = \int_X f(2-g)d\mu\,. \tag{1.4}$$

Given $0 \le F$ a measurable function on X, if we set $f = \frac{F}{2g-1}$ and $h = \frac{2-g}{2g-1}$ then $\int_X F d\nu = \int_X Fh\, dx$ which is the desired result, if we can prove that $1/2 \le g(x) \le 2$. Define the sets

$$E_n^1 = \left\{ x \in X \mid g(x) < \frac{1}{2} - \frac{1}{n} \right\} \quad \text{and} \quad E_n^2 = \left\{ x \in X \mid g(x) > 2 + \frac{1}{n} \right\}\,.$$

By substituting $f = \mathbf{1}_{E_n^j}$, $j = 1, 2$ in (1.4), we see that

$$\mu(E_n^j) = \nu(E_n^j) = 0 \text{ for } j = 1,2\,,$$

from which the bounds $1/2 \le g(x) \le 2$ hold. Also $\mu(\{x \in X \mid g(x) = 1/2\}) = 0$ and $\nu(\{x \in X \mid g(x) = 2\}) = 0$. Notice that if $F = 1$, then $h \in L^1(X)$. □

Remark 1.35. *The more general version of the Radon-Nikodym theorem. Suppose that $\mu(X) < \infty$, ν is a finite signed measure (by the Hahn decomposition, $\nu = \nu^- + \nu^+$) such that $\nu \ll \mu$; then, there exists $h \in L^1(X,\mu)$ such that $\int_X F\, d\nu = \int_X Fh\, d\mu$.*

Lemma 1.36 (Converse to Hölder's inequality)**.** *Let $\mu(X) < \infty$. Suppose that g is measurable and $fg \in L^1(X)$ for all simple functions f. If*

$$M(g) = \sup_{\|f\|_{L^p}=1} \left\{ \left| \int_X fg\, d\mu \right| \ : \ f \text{ is a simple function} \right\} < \infty\,, \tag{1.5}$$

then $g \in L^q(X)$, and $\|g\|_{L^q(X)} = M(g)$.

Proof. Let ϕ_n be a sequence of simple functions such that $\phi_n \to g$ a.e. and $|\phi_n| \le |g|$. Set

$$f_n = \frac{|\phi_n|^{q-1} \operatorname{sgn}(\phi_n)}{\|\phi_n\|_{L^q}^{q-1}}$$

so that $\|f_n\|_{L^p} = 1$ for $p = q/(q-1)$. By Fatou's lemma,

$$\|g\|_{L^q(X)} \le \liminf_{n\to\infty} \|\phi_n\|_{L^q(X)} = \liminf_{n\to\infty} \int_X |f_n \phi_n| d\mu\,.$$

Since $\phi_n \to g$ a.e., then

$$\|g\|_{L^q(X)} \le \liminf_{n\to\infty} \int_X |f_n\phi_n| d\mu \le \liminf_{n\to\infty} \int_X |f_n g| d\mu \le M(g)\,.$$

The reverse inequality is implied by Hölder's inequality. □

Proof of the $L^p(X)'$ representation theorem. We have already proven that there exists a natural inclusion $\iota : L^q(X) \to L^p(X)'$ which is an isometry. It remains to show that ι is surjective.

Let $\phi \in L^p(X)'$ and define a set function ν on measurable subsets $E \subset X$ by

$$\nu(E) = \int_X \mathbf{1}_E d\nu =: \phi(\mathbf{1}_E)\,.$$

Thus, if $\mu(E) = 0$, then $\nu(E) = 0$. Then

$$\int_X f\,d\nu =: \phi(f)$$

for all simple functions f, and by Lemma 1.23, this holds for all $f \in L^p(X)$. By the Radon-Nikodym theorem, there exists $0 \le g \in L^1(X)$ such that

$$\int_X f\,d\nu = \int_X fg\,d\mu \quad \forall\ f \in L^p(X)\,.$$

But

$$\phi(f) = \int_X f\,d\nu = \int_X fg\,d\mu \tag{1.6}$$

and since $\phi \in L^p(X)'$, then $M(g)$ given by (1.5) is finite, and by the converse to Hölder's inequality, $g \in L^q(X)$, and $\|\phi\|_{\text{op}} = M(g) = \|g\|_{L^q(X)}$. □

1.11. **Weak convergence.** The importance of the Representation Theorem 1.31 is in the use of the weak-* topology on the dual space $L^p(X)'$. Recall that for a Banach space $\mathbb{B}$ and for any sequence ϕ_j in the dual space $\mathbb{B}'$, $\phi_j \overset{*}{\rightharpoonup} \phi$ in $\mathbb{B}'$ weak-*, if $\langle \phi_j, f\rangle \to \langle \phi, f\rangle$ for each $f \in \mathbb{B}$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbb{B}'$ and $\mathbb{B}$.

Theorem 1.37 (Alaoglu's Lemma)**.** *If $\mathbb{B}$ is a Banach space, then the closed unit ball in $\mathbb{B}'$ is compact in the weak -* topology.*

Definition 1.38. *For $1 \le p < \infty$, a sequence $\{f_n\} \subset L^p(X)$ is said to weakly converge to $f \in L^p(X)$ if*

$$\int_X f_n(x)\phi(x)dx \to \int_X f(x)\phi(x)dx \quad \forall \phi \in L^q(X), q = \frac{p}{p-1}\,.$$

We denote this convergence by saying that $f_n \rightharpoonup f$ in $L^p(X)$ weakly.

Given that $L^p(X)$ is reflexive for $p \in (1,\infty)$, a simple corollary of Alaoglu's Lemma is the following

Theorem 1.39 (Weak compactness for L^p, $1 < p < \infty$)**.** *If $1 < p < \infty$ and $\{f_n\}$ is a bounded seequence in $L^p(X)$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightharpoonup f$ in $L^p(X)$ weakly.*

Definition 1.40. *A sequence $\{f_n\} \subset L^\infty(X)$ is said to converge weak-* to $f \in L^\infty(X)$ if*

$$\int_X f_n(x)\phi(x)dx \to \int_X f(x)\phi(x)dx \quad \forall \phi \in L^1(X)\,.$$

We denote this convergence by saying that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(X)$ weak-.*

Theorem 1.41 (Weak-* compactness for L^∞)**.** *If $\{f_n\}$ is a bounded sequence in $L^\infty(X)$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \overset{*}{\rightharpoonup} f$ in $L^\infty(X)$ weak-*.*

Lemma 1.42. *If $f_n \to f$ in $L^p(X)$, then $f_n \rightharpoonup f$ in $L^p(X)$.*

Proof. By Hölder's inequality,

$$\left|\int_X g(f_n - f)dx\right| \le \|f_n - f\|_{L^p}\|g\|_{L^q}\,.$$

□

Note that if f_n is weakly convergent, in general, this does not imply that f_n is strongly convergent.

Example 1.43. *If $p = 2$, let f_n denote any orthonormal sequence in $L^2(X)$. From Bessel's inequality*

$$\sum_{n=1}^{\infty} \left| \int_X f_n g dx \right| \le \|g\|^2_{L^2(X)} ,$$

we see that $f_n \rightharpoonup 0$ in $L^2(X)$.

This example shows that the map $f \mapsto \|f\|_{L^p}$ is continuous, but not weakly continuous. It is, however, weakly lower-semicontinuous.

Theorem 1.44. *If $f_n \rightharpoonup f$ weakly in $L^p(X)$, then $\|f\|_{L^p} \le \liminf_{n\to\infty} \|f_n\|_{L^p}$.*

Proof. As a consequence of Theorem 1.31,

$$\begin{aligned} \|f\|_{L^p(X)} &= \sup_{\|g\|_{L^q(X)}=1} \left| \int_X f g dx \right| = \sup_{\|g\|_{L^q(X)}=1} \lim_{n\to\infty} \left| \int_X f_n g dx \right| \\ &\le \sup_{\|g\|_{L^q(X)}=1} \liminf_{n\to\infty} \|f_n\|_{L^p} \|g\|_{L^q} . \end{aligned}$$

□

Theorem 1.45. *If $f_n \rightharpoonup f$ in $L^p(X)$, then f_n is bounded in $L^p(X)$.*

Theorem 1.46. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded. Suppose that*

$$\sup_n \|f_n\|_{L^p(\Omega)} \le M < \infty \quad \text{and} \quad f_n \to f \quad \text{a.e.}$$

If $1 < p < \infty$, then $f_n \rightharpoonup f$ in $L^p(\Omega)$.

Proof. Egoroff's theorem states that for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. By definition, $f_n \rightharpoonup f$ in $L^p(\Omega)$ for $p \in (1, \infty)$ if $\int_\Omega (f_n - f) g dx \to 0$ for all $g \in L^q(\Omega)$, $q = \frac{p}{p-1}$. We have the inequality

$$\int_\Omega (f_n - f) g dx \le \int_E |f_n - f|\, |g|\, dx + \int_{E^c} |f_n - f|\, |g|\, dx .$$

Choose $n \in \mathbb{N}$ sufficiently large, so that $|f_n(x) - f(x)| \le \delta$ for all $x \in E^c$. By Hölder's inequality,

$$\int_{E^c} |f_n - f|\, |g|\, dx \le \|f_n - f\|_{L^p(E^c)} \|g\|_{L^q(E^c)} \le \delta \mu(E^c) \|g\|_{L^q(\Omega)} \le C\delta$$

for a constant $C < \infty$.

By the Dominated Convergence Theorem, $\|f_n - f\|_{L^p(\Omega)} \le 2M$ so by Hölder's inequality, the integral over E is bounded by $2M\|g\|_{L^q(E)}$. Next, we use the fact that the integral is continuous with respect to the measure of the set over which the integral is taken. In particular, if $0 \le h$ is integrable, then for all $\delta > 0$, there exists $\epsilon > 0$ such that if the set E_ϵ has measure $\mu(E_\epsilon) < \epsilon$, then $\int_{E_\epsilon} h dx \le \delta$. To see this, either approximate h by simple functions, or use the Dominated Convergence theorem for the integral $\int_\Omega \mathbf{1}_{E_\epsilon}(x) h(x) dx$. □

Remark 1.47. *The proof of Theorem 1.46 does not work in the case that* $p = 1$*, as Hölder's inequality gives*

$$\int_E |f_n - f|\,|g|\,dx \leq \|f_n - f\|_{L^1(\Omega)} \|g\|_{L^\infty(E)}\,,$$

so we lose the smallness of the right-hand side.

Remark 1.48. *Suppose that* $E \subset X$ *is bounded and measurable, and let* $g = \mathbf{1}_E$*. If* $f_n \rightharpoonup f$ *in* $L^p(X)$*, then*

$$\int_E f_n(x)dx \to \int_E f(x)dx;$$

hence, if $f_n \rightharpoonup f$*, then the average of* f_n *converges to the average of* f *pointwise.*

1.12. **Integral operators.** If $u : \mathbb{R}^n \to \mathbb{R}$ satisfies certain integrability conditions, then we can define the operator K acting on the function u as follows:

$$Ku(x) = \int_{\mathbb{R}^n} k(x,y)u(y)dy\,,$$

where $k(x,y)$ is called the *integral kernel.*The mollification procedure, introduced in Definition 1.27, is one example of the use of integral operators; the Fourier transform is another.

Definition 1.49. *Let* $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ *denote the space of bounded linear operators from* $L^p(\mathbb{R}^n)$ *to itself. Using the Representation Theorem 1.31, the natural norm on* $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ *is given by*

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} = \sup_{\|f\|_{L^p}=1} \sup_{\|g\|_{L^q}=1} \left| \int_{\mathbb{R}^n} Kf(x)g(x)dx \right|.$$

Theorem 1.50. *Let* $1 \leq p < \infty$*,* $Ku(x) = \int_{\mathbb{R}^n} k(x,y)u(y)dy$*, and suppose that*

$$\int_{\mathbb{R}^n} |k(x,y)|dx \leq C_1 \ \forall y \in \mathbb{R}^n \ \text{and} \ \int_{\mathbb{R}^n} |k(x,y)|dy \leq C_2 \ \forall x \in \mathbb{R}^n\,,$$

where $0 < C_1, C_2 < \infty$*. Then* $K : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is bounded and*

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \leq C_1^{\frac{1}{p}} C_2^{\frac{p-1}{p}}\,.$$

In order to prove Theorem 1.50, we will need another well-known inequality.

Lemma 1.51 (Cauchy-Young Inequality)**.** *If* $\frac{1}{p} + \frac{1}{q} = 1$*, then for all* $a, b \geq 0$*,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Proof. Suppose that $a, b > 0$, otherwise the inequality trivially holds.

$$\begin{aligned} ab &= \exp(\log(ab)) = \exp(\log a + \log b) \quad (\text{since } a, b > 0) \\ &= \exp\left(\frac{1}{p}\log a^p + \frac{1}{q}\log b^q\right) \\ &\leq \frac{1}{p}\exp(\log a^p) + \frac{1}{q}\exp(\log b^q) \quad (\text{using the convexity of } \exp) \\ &= \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

where we have used the condition $\frac{1}{p} + \frac{1}{q} = 1$. $\square$

Lemma 1.52 (Cauchy-Young Inequality with δ)**.** *If* $\frac{1}{p}+\frac{1}{q}=1$*, then for all* $a,b\geq 0$*,*

$$ab \leq \delta\, a^p + C_\delta b^q\,, \qquad \delta > 0\,,$$

with $C_\delta = (\delta p)^{-q/p}q^{-1}$*.*

Proof. This is a trivial consequence of Lemma 1.51 by setting

$$ab = a\cdot(\delta p)^{1/p} \quad \frac{b}{(\delta p)^{1/p}}\,.$$

□

Proof of Theorem 1.50. According to Lemma 1.51, $|f(y)g(x)| \leq \frac{|f(y)|^p}{p}+\frac{|g(x)|^q}{q}$ so that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)f(y)g(x)dydx\right| \\
&\qquad\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{p}dx|f(y)|^p dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{q}dy|g(x)|^q dx \\
&\qquad\leq \frac{C_1}{p}\|f\|^p_{L^p} + \frac{C_2}{q}\|g\|^q_{L^q}\,.
\end{aligned}$$

To improve this bound, notice that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)f(y)g(x)dydx\right| \\
&\qquad\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{p}dx|tf(y)|^p dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{q}dy|t^{-1}g(x)|^q dx \\
&\qquad\leq \frac{C_1 t^p}{p}\|f\|^p_{L^p} + \frac{C_2 t^{-q}}{q}\|g\|^q_{L^q} =: F(t)\,.
\end{aligned}$$

Find the value of t for which $F(t)$ has a minimum to establish the desired bounded. □

Theorem 1.53 (Simple version of Young's inequality)**.** *Suppose that* $k\in L^1(\mathbb{R}^n)$ *and* $f\in L^p(\mathbb{R}^n)$*. Then*

$$\|k*f\|_{L^p}\leq \|k\|_{L^1}\|f\|_{L^p}\,.$$

Proof. Define

$$K_k(f) = k*f := \int_{\mathbb{R}^n} k(x-y)f(y)dy\,.$$

Let $C_1 = C_2 = \|k\|_{L^1(\mathbb{R}^n)}$. Then according to Theorem 1.50, $K_k : L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ and $\|K_k\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))}\leq C_1$. □

Theorem 1.50 can easily be generalized to the setting of integral operators $K : L^q(\mathbb{R}^n)\to L^r(\mathbb{R}^n)$ built with kernels $k\in L^p(\mathbb{R}^n)$ such that $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Such a generalization leads to

Theorem 1.54 (Young's inequality)**.** *Suppose that* $k\in L^p(\mathbb{R}^n)$ *and* $f\in L^q(\mathbb{R}^n)$*. Then*

$$\|k*f\|_{L^r}\leq \|k\|_{L^p}\|f\|_{L^q} \quad \text{for} \quad 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\,.$$

1.13. **Appendix 1: The Fubini and Tonelli Theorems.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ denote two fixed measure spaces. The product σ-algebra $\mathcal{A} \times \mathcal{B}$ of subsets of $X \times Y$ is defined by

$$\mathcal{A} \times \mathcal{B} = \{A \times B \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The set function $\mu \times \nu : \mathcal{A} \times \mathcal{B} \to [0, \infty]$ defined by

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

for each $A \times B \in \mathcal{A} \times \mathcal{B}$ is a measure.

Theorem 1.55 (Fubini)**.** *Let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-integrable function. Then both iterated integrals exist and*

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f \, d\mu d\nu = \int_X \int_Y f \, d\nu d\mu \, .$$

The existence of the iterated integrals is by no means enough to ensure that the function is integrable over the product space. As an example, let $X = Y = [0, 1]$ and $\mu = \nu = \lambda$ with λ the Lebesgue measure. Set

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} .$$

Then compute that

$$\int_0^1 f(x, y) dx dy = -\frac{\pi}{4}, \qquad \int_0^1 f(x, y) dy dx = \frac{\pi}{4} \, .$$

Fubini's theorem shows, of course, that f is not integrable over $[0, 1]^2$

There is a converse to Fubini's theorem, however, according to which the existence of one of the iterated integrals is sufficient for the integrability of the function over the product space. The theorem is known as Tonelli's theorem, and this result is often used.

Theorem 1.56 (Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ denote two σ-finite measure spaces, and let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-measurable function. If one of the iterated integrals $\int_X \int_Y |f| d\nu d\mu$ or $\int_Y \int_X |f| d\mu d\nu$ exists, then the function f is $\mu \times \nu$-integrable and hence, the other iterated integral exists and*

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f \, d\mu d\nu = \int_X \int_Y f \, d\nu d\mu \, .$$

2. Sobolev Spaces

2.1. **Weak derivatives.**

Definition 2.1 (Test functions)**.** *For $\Omega \subset \mathbb{R}^n$, set*

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) \mid \operatorname{spt}(u) \subset V \subset\subset \Omega\},$$

the smooth functions with compact support. Traditionally $\mathcal{D}(\Omega)$ is often used to denote $C_0^\infty(\Omega)$, and $\mathcal{D}(\Omega)$ is often referred to as the space of test functions.

For $u \in C^1(\mathbb{R})$, we can define $\frac{du}{dx}$ by the integration-by-parts formula; namely,

$$\int_{\mathbb{R}} \frac{du}{dx}(x) \phi(x) dx = -\int_{\mathbb{R}} u(x) \frac{d\phi}{dx}(x) dx \ \forall \phi \in C_0^\infty(\mathbb{R}) \, .$$

Notice, however, that the right-hand side is well-defined, whenever $u \in L^1_{\text{loc}}(\mathbb{R})$

Definition 2.2. *An element $\alpha \in \mathbb{Z}^n$ is called a multi-index. For such an $\alpha = (\alpha_1, ..., \alpha_n)$, we write $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_{\alpha_n}}$ and $|\alpha| = \alpha_1 + \cdots \alpha_n$.*

Example 2.3. *Let $n = 2$. If $|\alpha| = 0$, then $\alpha = (0, 0)$; if $|\alpha| = 1$, then $\alpha = (1, 0)$ or $\alpha = (0, 1)$. If $|\alpha| = 2$, then $\alpha = (1, 1)$.*

Definition 2.4 (Weak derivative)**.** *Suppose that $u \in L^1_{\text{loc}}(\Omega)$. Then $v^\alpha \in L^1_{\text{loc}}(\Omega)$ is called the α^{th} weak derivative of u, written $v^\alpha = D^\alpha u$, if*

$$\int_\Omega u(x)\, D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v^\alpha(x)\phi(x) dx \ \forall \phi \in C_0^\infty(\Omega)\,.$$

Example 2.5. *Let $n = 1$ and set $\Omega = (0, 2)$. Define the function*

$$u(x) = \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 \le x \le 2 \end{cases}\,.$$

Then the function

$$v(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

is the weak derivative of u. To see this, note that for $\phi \in C_0^\infty(0, 2)$,

$$\begin{aligned} \int_0^2 u(x)\frac{d\phi}{dx}(x)dx &= \int_0^1 x\frac{d\phi}{dx}(x)dx + \int_1^2 \frac{d\phi}{dx}(x)dx \\ &= -\int_0^1 \phi(x)dx + x\phi|_0^1 + \phi|_1^2 = -\int_0^1 \phi(x)dx \\ &= -\int_0^2 v(x)\phi(x)dx\,. \end{aligned}$$

Example 2.6. *Let $n = 1$ and set $\Omega = (0, 2)$. Define the function*

$$u(x) = \begin{cases} x, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}\,.$$

Then the weak derivative <u>does not</u> exist!

To prove this, assume for the sake of contradiction that there exists $v \in L^1_{\text{loc}}(\Omega)$ such that for all $\phi \in C_0^\infty(0, 2)$,

$$\int_0^2 v(x)\phi(x)dx = -\int_0^2 u(x)\frac{d\phi}{dx}(x)dx\,.$$

Then

$$\begin{aligned} \int_0^2 v(x)\phi(x)dx &= -\int_0^1 x\frac{d\phi}{dx}(x)dx - 2\int_1^2 \frac{d\phi}{dx}(x)dx \\ &= \int_0^1 \phi(x)dx - \phi(1) + 2\phi(1) \\ &= \int_0^1 \phi(x)dx + \phi(1)\,. \end{aligned}$$

Suppose that ϕ_j is a sequence in $C_0^\infty(0, 2)$ such that $\phi_j(1) = 1$ and $\phi_j(x) \to 0$ for $x \ne 1$. Then

$$1 = \phi_j(1) = \int_0^1 \phi_j(x)dx = \int_0^2 v(x)\phi_j(x)dx - \int_0^1 \phi_j(x)dx \to 0\,,$$

which provides the contradiction.

Definition 2.7. *For $p \in [1, \infty]$, define $W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid$ weak derivative exists $, Du \in L^p(\Omega)\}$, where Du is the weak derivative of u.*

Example 2.8. *Let $n = 1$ and set $\Omega = (0, 1)$. Define the function $f(x) = \sin(1/x)$. Then $u \in L^1(0, 1)$ and $\frac{du}{dx} = -\cos(1/x)/x^2 \in L^1_{\text{loc}}(0, 1)$, but $u \notin W^{1,p}(\Omega)$ for any p. In the case $p = 2$, we set $H^1(\Omega) = W^{1,p}(\Omega)$.*

Example 2.9. *Let $\Omega = B(0, 1) \subset \mathbb{R}^2$ and set $u(x) = |x|^{-\alpha}$. We want to determine the values of α for which $u \in H^1(\Omega)$.*

Since $|x|^{-\alpha} = \sum_{j=1}^3 (x_j x_j)^{-\alpha/2}$, then $\partial_{x_i} |x|^{-\alpha} = -\alpha |x|^{-\alpha-2} x_i$ is well-defined away from $x = 0$.

Step 1. *We show that $u \in L^1_{\text{loc}}(\Omega)$. To see this, note that $\int_\Omega |x|^{-\alpha} dx = \int_0^{2\pi} \int_0^1 r^{-\alpha} r dr d\theta < \infty$ whenever $\alpha < 2$.*

Step 2. *Set $v(x) = -\alpha |x|^{-\alpha-2} x_i$. We show that*

$$\int_{B(0,1)} u(x) D\phi(x) dx = -\int_{B(0,1)} v(x)\phi(x) dx \quad \forall \phi \in C_0^\infty(B(0, 1))\,.$$

To see this, let $\Omega_\delta = B(0, 1) - B(0, \delta)$, let n denote the inward-pointing unit normal to $\partial\Omega_\delta$. Integration by parts yields

$$\int_{\Omega_\delta} |x|^{-\alpha} D\phi(x) dx = \int_0^{2\pi} \delta^{-\alpha} \phi(x) n(x) \delta d\theta + \alpha \int_{\Omega_\delta} |x|^{-\alpha-2} x\, \phi(x) dx\,.$$

Since $\lim_{\delta \to 0} \delta^{1-\alpha} \int_0^{2\pi} \phi(x) n(x) d\theta = 0$ if $\alpha < 1$, we see that

$$\lim_{\delta \to 0} \int_{\Omega_\delta} |x|^{-\alpha} D\phi(x) dx = \lim_{\delta \to 0} \alpha \int_{\Omega_\delta} |x|^{-\alpha-2} x\, \phi(x) dx$$

Since $\int_0^{2\pi} \int_0^1 r^{-\alpha-1} r dr d\theta < \infty$ if $\alpha < 1$, the Dominated Convergence Theorem shows that v is the weak derivative of u.

Step 3. *$v \in L^2(\Omega)$, whenever $\int_0^{2\pi} \int_0^1 r^{-2\alpha-2} r dr d\theta < \infty$ which holds if $\alpha < 0$.*

Remark 2.10. *Note that if the weak derivative exists, it is unique. To see this, suppose that both v_1 and v_2 are the weak derivative of u on Ω. Then $\int_\Omega (v_1 - v_2)\phi dx = 0$ for all $\phi \in C_0^\infty(\Omega)$, so that $v_1 = v_2$ a.e.*

2.2. **Definition of Sobolev Spaces.**

Definition 2.11. *For integers $k \geq 0$ and $1 \leq p \leq \infty$,*

$$W^{k,p}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) \mid D^\alpha u \text{ exists and is in } L^p(\Omega) \text{ for } |\alpha| \leq k\}.$$

Definition 2.12. *For $u \in W^{k,p}(\Omega)$ define*

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty\,,$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}\,.$$

The function $\| \cdot \|_{W^{k,p}(\Omega)}$ is clearly a norm since it is a finite sum of L^p norms.

Definition 2.13. *A sequence $u_j \to u$ in $W^{k,p}(\Omega)$ if $\lim_{j \to \infty} \|u_j - u\|_{W^{k,p}(\Omega)} = 0$.*

Theorem 2.14. *$W^{k,p}(\Omega)$ is a Banach space.*

Proof. Let u_j denote a Cauchy sequence in $W^{k,p}(\Omega)$. It follows that for all $|\alpha| \le k$, $D^\alpha u_j$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space (see Theorem 1.19), for each α there exists $u^\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_j \to u^\alpha \text{ in } L^p(\Omega)\,.$$

When $\alpha = (0, ..., 0)$ we set $u := u^{(0,\dots,0)}$ so that $u_j \to u$ in $L^p(\Omega)$. We must show that $u^\alpha = D^\alpha u$.

For each $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega u D^\alpha \phi dx &= \lim_{j\to\infty} \int_\Omega u_j D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \lim_{j\to\infty} \int_\Omega D^\alpha u_j \phi dx \\ &= (-1)^{|\alpha|} \int_\Omega u^\alpha \phi dx\,;\end{aligned}$$

thus, $u^\alpha = D^\alpha u$ and hence $D^\alpha u_j \to D^\alpha u$ in $L^p(\Omega)$ for each $|\alpha| \le k$, which shows that $u_j \to u$ in $W^{k,p}(\Omega)$. □

Definition 2.15. *For integers $k \ge 0$ and $p = 2$, we define*

$$H^k(\Omega) = W^{k,2}(\Omega)\,.$$

$H^k(\Omega)$ is a Hilbert space with inner-product $(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$.

2.3. **A simple version of the Sobolev embedding theorem.** For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, we say that $\mathbb{B}_1$ is embedded in $\mathbb{B}_2$ if $\|u\|_{\mathbb{B}_2} \le C\|u\|_{\mathbb{B}_1}$ for some constant C and for $u \in \mathbb{B}_1$. We wish to determine which Sobolev spaces $W^{k,p}(\Omega)$ can be embedded in the space of continuous functions. To motivate the type of analysis that is to be employed, we study a special case.

Theorem 2.16 (Sobolev embedding in 2-D)**.** *For $kp \ge 2$,*

$$\max_{x\in\mathbb{R}^2} |u(x)| \le C\|u\|_{W^{k,p}(\mathbb{R}^2)} \quad \forall u \in C_0^\infty(\Omega)\,. \tag{2.1}$$

Proof. Given $u \in C_0^\infty(\Omega)$, we prove that for all $x \in \text{spt}(u)$,

$$|u(x)| \le C\|D^\alpha u(x)\|_{L^2(\Omega)} \quad \forall |\alpha| \le k\,.$$

By choosing a coordinate system centered about x, we can assume that $x = 0$; thus, it suffices to prove that

$$|u(0)| \le C\|D^\alpha u(x)\|_{L^2(\Omega)} \quad \forall |\alpha| \le k\,.$$

Let $0 \le g \in C^\infty([0, \infty))$ such that $g(x) = 1$ for $x \in [0, \frac{1}{2}]$ and $g(x) = 0$ for $x \in [\frac{3}{4}, \infty)$.

By the fundamental theorem of calculus,

$$\begin{aligned}u(0) &= -\int_0^1 \partial_r[g(r)u(r,\theta)]dr = -\int_0^1 \partial_r(r)\,\partial_r[g(r)u(r,\theta)]dr \\ &= \int_0^1 r\,\partial_r^2[g(r)u(r,\theta)]dr \\ &= \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-1}\,\partial_r^k[g(r)u(r,\theta)]dr = \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-2}\,\partial_r^k[g(r)u(r,\theta)]rdr\end{aligned}$$

Integrating both sides from 0 to 2π, we see that

$$u(0) = \frac{(-1)^k}{2\pi(k-1)!}\int_0^{2\pi}\int_0^1 r^{k-2}\,\partial_r^k[g(r)u(r,\theta)]rdrd\theta\,.$$

The change of variables from Cartesian to polar coordinates is given by

$$x(r,\theta) = r\cos\theta\,,\quad y(r,\theta) = r\sin\theta\,.$$

By the chain-rule,

$$\begin{aligned}\partial_r u(x(r,\theta),y(r,\theta)) &= \partial_x u\cos\theta + \partial_y u\sin\theta\,,\\ \partial_r^2 u(x(r,\theta),y(r,\theta)) &= \partial_x^2 u\cos^2\theta + 2\partial_{xy}^2 u\cos\theta\sin\theta + \partial_y^2 u\sin^2\theta\\ &\ \ \vdots\end{aligned}$$

It follows that $\partial_r^k = \sum_{|\alpha|\le k} a_\alpha(\theta)D^\alpha$ so that

$$\begin{aligned}u(0) &= \frac{(-1)^k}{2\pi(k-1)!}\int_{B(0,1)} r^{k-2}\sum_{|\alpha|\le k} a_\alpha(\theta)D^\alpha[g(r)u(x)]dx\\ &\le \|r^{k-2}\|_{L^q(B(0,1))}\sum_{|\alpha|\le k}\|D^\alpha(gu)\|_{L^p(B(0,1))}\\ &\le C\left(\int_0^1 r^{\frac{p(k-2)}{p-1}}rdr\right)^{\frac{p-1}{p}}\|u\|_{W^{k,p}(\mathbb{R}^2)}\,.\end{aligned}$$

Hence, we require $\frac{p(k-2)}{p-1}+1>-1$ or $kp>2$. □

2.4. Approximation of $W^{k,p}(\Omega)$ by smooth functions.

Recall that $\Omega_\epsilon = \{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)>\epsilon\}$.

Theorem 2.17. *For integers $k\ge 0$ and $1\le p<\infty$, let*

$$u^\epsilon = \eta_\epsilon * u \text{ in } \Omega_\epsilon\,,$$

where η_ϵ is the standard mollifier defined in Definition 1.25. Then
- (A) *$u^\epsilon\in C^\infty(\Omega_\epsilon)$ for each $\epsilon>0$, and*
- (B) *$u^\epsilon\to u$ in $W^{k,p}_{\text{loc}}(\Omega)$ as $\epsilon\to 0$.*

Definition 2.18. *A sequence $u_j\to u$ in $W^{k,p}_{\text{loc}}(\Omega)$ if $u_j\to u$ in $W^{k,p}(\tilde\Omega)$ for each $\tilde\Omega\subset\subset\Omega$.*

Proof of Theorem 2.17. Theorem 1.28 proves part (A). Next, let v^α denote the the αth weak partial derivative of u. To prove part (B), we show that $D^\alpha u^\epsilon = \eta_\epsilon * v^\alpha$ in Ω_ϵ. For $x\in\Omega_\epsilon$,

$$\begin{aligned}D^\alpha u^\epsilon(x) &= D^\alpha\int_\Omega \eta_\epsilon(x-y)u(y)dy\\ &= \int_\Omega D_x^\alpha\eta_\epsilon(x-y)u(y)dy\\ &= (-1)^{|\alpha|}\int_\Omega D_y^\alpha\eta_\epsilon(x-y)u(y)dy\\ &= \int_\Omega \eta_\epsilon(x-y)v^\alpha(y)dy = (\eta_\epsilon * v^\alpha)(x)\,.\end{aligned}$$

By part (D) of Theorem 1.28, $D^\alpha u^\epsilon\to v^\alpha$ in $L^p_{\text{loc}}(\Omega)$. □

It is possible to refine the above *interior* approximation result all the way to the boundary of Ω. We record the following theorem without proof.

Theorem 2.19. *Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth, open, bounded subset, and that $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$ and integers $k \ge 0$. Then there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that*

$$u_j \to u \quad in \quad W^{k,p}(\Omega)\,.$$

It follows that the inequality (2.1) holds for all $u \in W^{k,p}(\mathbb{R}^2)$.

2.5. **Hölder Spaces.** Recall that for $\Omega \subset \mathbb{R}^n$ open and smooth, the class of Lipschitz functions $u : \Omega \to \mathbb{R}$ satisfies the estimate

$$|u(x) - u(y)| \le C|x - y| \quad \forall x, y \in \Omega$$

for some constant C.

Definition 2.20 (Classical derivative)**.** *A function $u : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ if there exists $f : \Omega \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ such that*

$$\frac{|u(x) - u(y) - f(x)\cdot(x - y)|}{|x - y|} \to 0\,.$$

We call $f(x)$ the gradient of $u(x)$, and denote it by $Du(x)$.

Definition 2.21. *If $u : \Omega \to \mathbb{R}$ is bounded and continuous, then*

$$\|u\|_{C^0(\overline{\Omega})} = \max_{x\in\Omega} |u(x)|\,.$$

If in addition u has a continuous and bounded derivative, then

$$\|u\|_{C^1(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + \|Du\|_{C^0(\overline{\Omega})}\,.$$

The Hölder spaces *interpolate* between $C^0(\overline{\Omega})$ and $C^1(\overline{\Omega})$.

Definition 2.22. *For $0 < \gamma \le 1$, the space $C^{0,\gamma}(\overline{\Omega})$ consists of those functions for which*

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})} < \infty\,,$$

where the γth Hölder semi-norm $[u]_{C^{0,\gamma}(\overline{\Omega})}$ is defined as

$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \max_{\substack{x,y\in\Omega\\ x\neq y}} \left(\frac{|u(x) - u(y)|}{|x - y|} \right)\,.$$

The space $C^{0,\gamma}(\overline{\Omega})$ is a Banach space.

2.6. **Morrey's inequality.** We can now offer a refinement and extension of the simple version of the Sobolev Embedding Theorem 2.16.

Theorem 2.23 (Morrey's inequality)**.** *For $n < p \le \infty$, let $B(x, r) \subset \mathbb{R}^n$ and let $y \in B(x, r)$. Then*

$$|u(x) - u(y)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,2r))} \forall u \in C^1(\mathbb{R}^n)\,.$$

Notation 2.24 (Averaging)**.** *Let $B(0,1) \subset \mathbb{R}^n$. The volume of $B(0,1)$ is given by $\alpha_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ and the surface area is $|\mathbb{S}^{n-1}| = n\alpha_n$. We define*

$$\fint_{B(x,r)} f(y)dy = \frac{1}{\alpha_n r^n} \int_{B(x,r)} f(y)dy$$
$$\fint_{\partial B(x,r)} f(y)dS = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} f(y)dS\,.$$

Lemma 2.25. *For $B(x,r) \subset \mathbb{R}^n$, $y \in B(x,r)$ and $u \in C^1(\overline{B(x,r)})$,*

$$\fint_{B(x,r)} |u(y) - u(x)|dy \le C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,.$$

Proof. For some $0 < s < r$, let $y = x + s\omega$ where $\omega \in \mathbb{S}^{n-1} = \partial B(0,1)$. By the fundamental theorem of calculus, for $0 < s < r$,

$$\begin{aligned} u(x+s\omega) - u(x) &= \int_0^s \frac{d}{dt} u(x+t\omega)dt \\ &= \int_0^s Du(x+t\omega)\,\omega dt\,. \end{aligned}$$

Since $|\omega| = 1$, it follows that

$$|u(x+s\omega) - u(x)| \le \int_0^s |Du(x+t\omega)|dt\,.$$

Thus, integrating over $\mathbb{S}^{n-1}$ yields

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |u(x+s\omega) - u(x)|d\omega &\le \int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+t\omega)|d\omega dt \\ &\le \int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+t\omega)| \frac{t^{n-1}}{t^{n-1}} d\omega dt \\ &= \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,, \end{aligned}$$

where we have set $y = x + t\omega$ for the last equality.

Multipling the above inequality by s^{n-1} and integrating s from 0 to r shows that

$$\begin{aligned} \int_0^r \int_{\mathbb{S}^{n-1}} |u(x+s\omega) - u(x)|d\omega s^{n-1} ds &\le \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \\ &\le C\alpha_n r^n \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,, \end{aligned}$$

which proves the lemma. ∎

Proof of Theorem 2.23. Assume first that $u \in C^1(\overline{B(x,2r)})$. Let $D = B(x,r) \cap B(y,r)$ and set $r = |x-y|$. Then

$$\begin{aligned}
|u(x)-u(y)| &= \fint_D |u(x)-u(y)|dz \\
&\le \fint_D |u(x)-u(z)|dz + \fint_D |u(y)-u(z)|dz \\
&\le C\fint_{B(x,r)} |u(x)-u(z)|dz + C\fint_{B(y,r)} |u(y)-u(z)|dz \\
&\le C\fint_{B(x,2r)} |u(x)-u(z)|dz\,.
\end{aligned}$$

Thus, by Lemma 2.25,

$$|u(x)-u(y)| \le C\int_{B(x,2r)} |x-z|^{1-n}|Du(z)|dz$$

and by Hölder's inequality,

$$|u(x)-u(y)| \le C\left(\int_{B(0,2r)} s^{\frac{p(1-n)}{p-1}} s^{n-1} ds d\omega\right)^{\frac{p-1}{p}} \left(\int_{B(x,2r)} |Du(z)|^p dz\right)^{\frac{1}{p}}$$

□

Morrey's inequality implies the following embedding theorem.

Theorem 2.26 (Sobolev embedding theorem for $k=1$)**.** *There exists a constant $C = C(p,n)$ such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n)\,.$$

Proof. First assume that $u \in C^1(\mathbb{R}^n)$. Given Morrey's inequality, it suffices to show that $\max|u| \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$. Using Lemma 2.25, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
|u(x)| &\le \fint_{B(x,1)} |u(x)-u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\
&\le C\int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + C\|u\|_{L^p(\mathbb{R}^n)} \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^n)}\,,
\end{aligned}$$

the last inequality following whenever $p > n$.

Thus,

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C^1(\mathbb{R}^n)\,. \tag{2.2}$$

By the density of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$, there is a sequence $u_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$u_j \to u \in W^{1,p}(\mathbb{R}^n)\,.$$

By (2.2), for $j,k \in \mathbb{N}$,

$$\|u_j - u_k\|C^{0,1-\frac{n}{p}}(\mathbb{R}^n) \le C\|u_j-u_k\|_{W^{1,p}(\mathbb{R}^n)}\,.$$

Since $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ is a Banach space, there exists a $U \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ such that

$$u_j \to U \;\text{ in }\; C^{0,1-\frac{n}{p}}(\mathbb{R}^n)\,.$$

It follows that $U = u$ a.e. in Ω. By the continuity of norms with respect to strong convergence, we see that

$$\|U\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

which completes the proof. $\square$

Remark 2.27. *By approximation, Morrey's inequality holds for all $u \in W^{1,p}(B(x,2r))$ for $n < p < \infty$. You are asked to prove this.*

As a consequence of Morrey's inequality, we extract information about the classical differentiability properties of weak derivatives.

Theorem 2.28 (Differentiability a.e.)**.** *If $\Omega \subset \mathbb{R}^n$, $n < p \leq \infty$ and $u \in W^{1,p}_{\text{loc}}(\Omega)$, then u is differentiable a.e. in Ω, and its gradient equals its weak gradient almost everywhere.*

Proof. We first restrict $n < p < \infty$. By Lebesgue's differentiation theorem, for almost every $x \in \Omega$,

$$\lim_{r\to 0} \fint_{B(x,r)} |Du(x) - Du(z)|^p dz = 0\,. \tag{2.3}$$

Fix $x \in \Omega$ for which (2.3) holds, and define the function

$$w_x(y) = u(y) - u(x) - Du(x)\cdot(y-x)\,.$$

Notice that $w_x(x) = 0$ and that

$$D_y w_x(y) = Du(y) - Du(x)\,.$$

Set $r = |x-y|$. An application of Morrey's inequality then yields the estimate

$$\begin{aligned}
|u(y) - u(x) - Du(x)\cdot(y-x)| &= |w_x(y) - w_x(x)| \\
&\leq C\int_{B(x,2r)} \frac{|D_z w_x(z)|}{|x-z|^{n-1}} dz \\
&= C\int_{B(x,2r)} \frac{|Du(z) - Du(x)|}{|x-z|^{n-1}} dz \\
&\leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,r)} |Du(z) - Du(x)|^p dz\right)^{\frac{1}{p}} \\
&\leq Cr \left(\fint_{B(x,r)} |Du(z) - Du(x)|^p dz\right)^{\frac{1}{p}} \\
&= o(r) \text{ as } r \to 0\,.
\end{aligned}$$

The case that $p = \infty$ follows from the inclusion $W^{1,\infty}_{\text{loc}}(\Omega) \subset W^{1,p}_{\text{loc}}(\Omega)$ for all $1 \leq p < \infty$. $\square$

2.7. **The Gagliardo-Nirenberg-Sobolev inequality.** In the previous section, we considered the embedding for the case that $p > n$.

Theorem 2.29 (Gagliardo-Nirenberg inequality)**.** *For $1 \leq p < n$, set $p^* = \frac{np}{n-p}$. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{p,n}\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n)\,.$$

Proof for the case $n = 2$. Suppose first that $p = 1$ in which case $p^* = 2$, and we must prove that

$$\|u\|_{L^2(\mathbb{R}^2)} \le C\|Du\|_{L^1(\mathbb{R}^2)} \quad \forall u \in C^1_0(\mathbb{R}^2)\,. \tag{2.4}$$

Since u has compact support, by the fundamental theorem of calculus,

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(y_1, x_2)dy_1 = \int_{-\infty}^{x_2} \partial_2 u(x_1, y_2)dy_2$$

so that

$$|u(x_1, x_2)| \le \int_{-\infty}^{\infty} |\partial_1 u(y_1, x_2)|dy_1 \le \int_{-\infty}^{\infty} |Du(y_1, x_2)|dy_1$$

and

$$|u(x_1, x_2)| \le \int_{-\infty}^{\infty} |\partial_2 u(x_1, y_2)|dy_2 \le \int_{-\infty}^{\infty} |Du(x_1, y_2)|dy_2\,.$$

Hence, it follows that

$$|u(x_1, x_2)|^2 \le \int_{-\infty}^{\infty} |Du(y_1, x_2)|dy_1 \int_{-\infty}^{\infty} |Du(x_1, y_2)|dy_2$$

Integrating over $\mathbb{R}^2$, we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1, x_2)|^2 dx_1 dx_2 \\
&\qquad \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du(y_1, x_2)|dy_1 \int_{-\infty}^{\infty} |Du(x_1, y_2)|dy_2 \right) dx_1 dx_2 \\
&\qquad \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1, x_2)|dx_1 dx_2 \right)^2
\end{aligned}$$

which is (2.4).

Next, if $1 \le p < 2$, substitute $|u|^\gamma$ for u in (2.4) to find that

$$\begin{aligned}
\left(\int_{\mathbb{R}^2} |u|^{2\gamma} dx \right)^{\frac{1}{2}} &\le C\gamma \int_{\mathbb{R}^2} |u|^{\gamma-1} |Du| dx \\
&\le C\gamma \|Du\|_{L^p(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} |u|^{\frac{p(\gamma-1)}{p-1}} dx \right)^{\frac{p-1}{p}}
\end{aligned}$$

Choose γ so that $2\gamma = \frac{p(\gamma-1)}{p-1}$; hence, $\gamma = \frac{p}{2-p}$, and

$$\left(\int_{\mathbb{R}^2} |u|^{\frac{2p}{2-p}} dx \right)^{\frac{2-p}{2p}} \le C\gamma \|Du\|_{L^p(\mathbb{R}^2)}\,,$$

so that

$$\|u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^n)} \le C_{p,n} \|Du\|_{L^p(\mathbb{R}^n)} \tag{2.5}$$

for all $u \in C^1_0(\mathbb{R}^2)$.

Since $C^\infty_0(\mathbb{R}^2)$ is dense in $W^{1,p}(\mathbb{R}^2)$, there exists a sequence $u_j \in C^\infty_0(\mathbb{R}^2)$ such that

$$u_j \to u \ \text{ in } \ W^{1,p}(\mathbb{R}^2)\,.$$

Hence, by (2.5), for all $j, k \in \mathbb{N}$,

$$\|u_j - u_k\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^n)} \le C_{p,n} \|Du_j - Du_k\|_{L^p(\mathbb{R}^n)}$$

so there exists $U \in L^{\frac{2p}{2-p}}(\mathbb{R}^n)$ such that

$$u_j \to U \quad \text{in} \quad L^{\frac{2p}{2-p}}(\mathbb{R}^n)\,.$$

Hence $U = u$ a.e. in $\mathbb{R}^2$, and by continuity of the norms, (2.5) holds for all $u \in W^{1,p}(\mathbb{R}^2)$. □

It is common to employ the Gagliardo-Nirenberg inequality for the case that $p = 2$; as stated, the inequality is not well-defined in dimension two, but in fact, we have the following theorem.

Theorem 2.30. *Suppose that $u \in H^1(\mathbb{R}^2)$. Then for all $1 \le q < \infty$,*

$$\|u\|_{L^q(\mathbb{R}^2)} \le C\sqrt{q}\|u\|_{H^1(\mathbb{R}^2)}\,.$$

Proof. Let x and y be points in $\mathbb{R}^2$, and write $r = |x-y|$. Let $\theta \in \mathbb{S}^1$. Introduce spherical coordinates (r,θ) with origin at x, and let g be the same cut-off function that was used in the proof of Theorem 2.16. Define $U := g(r)u(r,\theta)$. Then

$$u(x) = -\int_0^1 \frac{\partial U}{\partial r}(r,\theta)dr - \int_0^1 |x-y|^{-1}\frac{\partial U}{\partial r}(r,\theta)rdr$$

and

$$|u(x)| \le \int_0^1 |x-y|^{-1}|DU(r,\theta)|rdr\,.$$

Integrating over $\mathbb{S}^1$, we obtain:

$$|u(x)| \le \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{1}_{B(x,1)}|x-y|^{-1}|DU(y)|dy := K * |DU|\,,$$

where the integral kernel $K(x) = \frac{1}{2\pi}\mathbf{1}_{B(0,1)}|x|^{-1}$.

Using Young's inequality from Theorem 1.54, we obtain the estimate

$$\|K * f\|_{L^q(\mathbb{R}^2)} \le \|K\|_{L^k(\mathbb{R}^2)}\|f\|_{L^2(\mathbb{R}^2)} \quad \text{for} \quad \frac{1}{k} = \frac{1}{q} - \frac{1}{2} + 1\,. \tag{2.6}$$

Using the inequality (2.6) with $f = |DU|$, we see that

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^2)} &\le C\|DU\|_{L^2(\mathbb{R}^2)}\left[\int_{B(0,1)} |y|^{-k}dy\right]^{\frac{1}{k}} \\
&\le C\|DU\|_{L^2(\mathbb{R}^2)}\left[\int_0^1 r^{1-k}dr\right]^{\frac{1}{k}} \\
&= C\|u\|_{H^1(\mathbb{R}^2)}\left[\frac{q+2}{4}\right]^{\frac{1}{k}}\,.
\end{aligned}$$

When $q \to \infty$, $\frac{1}{k} \to \frac{1}{2}$, so

$$\|u\|_{L^q(\mathbb{R}^2)} \le Cq^{\frac{1}{2}}\|u\|_{H^1(\mathbb{R}^2)}\,.$$

□

Evidently, it is not possible to obtain the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{W^{1,n}(\mathbb{R}^n)}$ with a constant $C < \infty$. The following provides an example of a function in this borderline situation.

Example 2.31. *Let $\Omega \subset \mathbb{R}^2$ denote the open unit ball in $\mathbb{R}^2$. The unbounded function $u = \log\log\left(1+\frac{1}{|x|}\right)$ belongs to $H^1(B(0,1))$.*

First, note that

$$\int_\Omega |u(x)|^2 dx = \int_0^{2\pi}\int_0^1 \left[\log\log\left(1+\frac{1}{r}\right)\right]^2 r dr d\theta\,.$$

The only potential singularity of the integrand occurs at $r=0$, but according to L'Hospital's rule,

$$\lim_{r\to 0} r\left[\log\log\left(1+\frac{1}{r}\right)\right]^2 = 0, \tag{2.7}$$

so the integrand is continuous and hence $u \in L^2(\Omega)$.

In order to compute the partial derivatives of u, note that

$$\frac{\partial}{\partial x_j}|x| = \frac{x_j}{|x|}\,, \text{ and } \frac{d}{dz}|f(z)| = \frac{f(x)\frac{df}{dz}}{|f(z)|}\,,$$

where $f:\mathbb{R}\to\mathbb{R}$ is differentiable. It follows that for x away from the origin,

$$Du(x) = \frac{-x}{\log(1+\frac{1}{|x|})(|x|+1)|x|^2}\,, \quad (x\neq 0)\,.$$

Let $\phi \in C_0^\infty(\Omega)$ and fix $\epsilon > 0$. Then

$$\int_{\Omega - B_\epsilon(0)} u(x)\frac{\partial\phi}{\partial x_i}(x)dx = -\int_{\Omega-B(0,\epsilon)}\frac{\partial u}{\partial x_i}(x)\phi(x)dx + \int_{\partial B(0,\epsilon)} u\phi N_i dS\,,$$

where $N=(N_1,...,N_n)$ denotes the inward-pointing unit normal on the curve $\partial B(0,\epsilon)$, so that $N\,dS = \epsilon(\cos\theta,\sin\theta)d\theta$. It follows that

$$\int_{\Omega-B_\epsilon(0)} u(x)D\phi(x)dx = -\int_{\Omega-B_\epsilon(0)} Du(x)\phi(x)dx - \int_0^{2\pi}\epsilon(\cos\theta,\sin\theta)\,\log\log\left(1+\frac{1}{\epsilon}\right)\phi(\epsilon,\theta)d\theta\,. \tag{2.8}$$

We claim that $Du \in L^2(\Omega)$ (and hence also in $L^1(\Omega)$), for

$$\begin{aligned}\int_\Omega |Du(x)|^2dx &= \int_0^{2\pi}\int_0^1 \frac{1}{r(r+1)^2\left[\log\left(1+\frac{1}{r}\right)\right]^2}drd\theta \\ &\leq \pi\int_0^{1/2}\frac{1}{r(\log r)^2}dr + \pi\int_{1/2}^1 \frac{1}{r(r+1)^2\left[\log\left(1+\frac{1}{r}\right)\right]^2}dr\end{aligned}$$

where we use the inequality $\log(1+\frac{1}{r}) \geq \log\frac{1}{r} = -\log r \geq 0$ for $0 \leq r \leq 1$. The second integral on the right-hand side is clearly bounded, while

$$\int_0^{1/2}\frac{1}{r(\log r)^2}dr = \int_{-\infty}^{-\log 2}\frac{1}{t^2e^t}e^tdt = \int_{-\infty}^{-\log 2}\frac{1}{x^2}dx < \infty\,,$$

so that $Du \in L^2(\Omega)$. Letting $\epsilon \to 0$ in (2.8) and using (2.7) for the boundary integral, by the Dominated Convergence Theorem, we conclude that

$$\int_\Omega u(x)D\phi(x)dx = -\int_\Omega Du(x)\phi(x)dx \quad \forall \phi \in C_0^\infty(\Omega)\,.$$

2.8. **Local coordinates near $\partial\Omega$.** Let $\Omega \subset \mathbb{R}^n$ denote an open, bounded subset with C^1 boundary, and let $\{U_l\}_{l=1}^K$ denote an open covering of $\partial\Omega$, such that for each $l \in \{1, 2, ..., K\}$, with

$$\begin{aligned}
&\mathcal{V}_l = B(0, r_l), \text{ denoting the open ball of radius } r_l \text{ centered at the origin and,}\\
&\mathcal{V}_l^+ = \mathcal{V}_l \cap \{x_n > 0\},\\
&\mathcal{V}_l^- = \mathcal{V}_l \cap \{x_n < 0\},
\end{aligned}$$

there exist C^1-class *charts* θ_l which satisfy

$$\begin{aligned}
\theta_l : \mathcal{V}_l \to U_l \ \text{ is a } C^1 \text{ diffeomorphism},\\
\theta_l(\mathcal{V}_l^+) = U_l \cap \Omega,\\
\theta_l(\mathcal{V}_l \cap \{x_n = 0\}) = U_l \cap \partial\Omega.
\end{aligned} \tag{2.9}$$

2.9. **Sobolev extensions and traces.** Let $\Omega \subset \mathbb{R}^n$ denote an open, bounded domain with C^1 boundary.

Theorem 2.32. *Suppose that $\tilde{\Omega} \subset \mathbb{R}^n$ is a bounded and open domain such that $\Omega \subset\subset \tilde{\Omega}$. Then for $1 \le p \le \infty$, there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that for all $u \in W^{1,p}(\Omega)$,

(1) *$Eu = u$ a.e. in Ω;*
(2) spt$(u) \subset \tilde{\Omega}$*;*
(3) *$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$ for a constant $C = C(p, \Omega, \tilde{\Omega})$.*

Theorem 2.33. *For $1 \le p < \infty$, there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\Omega)$$

such that for all $u \in W^{1,p}(\Omega)$

(1) *$Tu = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cup C^0(\overline{\Omega})$;*
(2) *$\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$ for a constant $C = C(p, \Omega)$.*

Proof. Suppose that $u \in C^1(\overline{\Omega})$, $z \in \partial\Omega$, and that $\partial\Omega$ is locally flat near z. In particular, for $r > 0$ sufficiently small, $B(z, r) \cup \partial\Omega \subset \{x_n = 0\}$. Let $0 \le \xi \in C_0^\infty(B(z, r)$ such that $\xi = 1$ on $B(z, r/2)$. Set $\Gamma = \partial\Omega \cup B(z, r/2)$, $B^+(z, r) = B(z, r) \cup \Omega$, and let $dx_h = dx_1 \cdots dx_{n-1}$. Then

$$\begin{aligned}
\int_\Gamma |u|^p dx_h &\le \int_{\{x_n=0\}} \xi |u|^p dx_h\\
&= -\int_{B^+(z,r)} \frac{\partial}{\partial x_n}(\xi |u|^p) dx\\
&\le -\int_{B^+(z,r)} \frac{\partial \xi}{\partial x_n} |u|^p dx - p\int_{B^+(z,2\delta)} \xi |u|^{p-2} u \frac{\partial u}{\partial x_n} dx\\
&\le C\int_{B^+(z,r)} |u|^p dx + C \| |u|^{p-1} \|_{L^{\frac{p}{p-1}}(B^+(z,r))} \left\| \frac{\partial u}{\partial x_n} \right\|_{L^p(B^+(z,r))}\\
&\le C\int_{B^+(z,r)} (|u|^p + |Du|^p) dx .
\end{aligned} \tag{2.10}$$

On the other hand, if the boundary is not locally flat near $z \in \partial\Omega$, then we use a C^1 diffeomorphism to locally *straighten the boundary.* More specifically, suppose that $z \in \partial\Omega \cup U_l$ for some $l \in \{1, ..., K\}$ and consider the C^1 chart θ_l defined in (2.9). Define the function $U = u \circ \theta_l$; then $U : V_l^+ \to \mathbf{R}$. Setting $\Gamma = V_l \cup \{x_n = 0\|$, we see from the inequality (2.10), that

$$\int_\Gamma |U|^p dx_h \leq C_l \int_{V_l^+} (|U|^p + |DU|^p) dx \,.$$

Using the fact that $D\theta_l$ is bounded and continuous on V_l^+, the change of variables formula shows that

$$\int_{U_l \cup \partial\Omega} |u|^p dS \leq C_l \int_{U_l^+} (|u|^p + |Du|^p) dx \,.$$

Summing over all $l \in \{1, ..., K\}$ shows that

$$\int_{\partial\Omega} |u|^p dS \leq C \int_\Omega (|u|^p + |Du|^p) dx \,. \tag{2.11}$$

The inequality (2.11) holds for all $u \in C^1(\overline{\Omega})$. According to Theorem 2.19, for $u \in W^{1,p}(\Omega)$ there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that $u_j \to u$ in $W^{1,p}(\Omega)$. By inequality (2.11),

$$\|Tu_k - Tu_j\|_{L^p(\partial\Omega)} \leq C\|u_k - u_j\|_{W^{1,p}(\Omega)} \,,$$

so that Tu_j is Cauchy in $L^p(\partial\Omega)$, and hence a limit exists in $L^p(\partial\Omega)$ We define the trace operator T as this limit:

$$\lim_{j\to 0} \|Tu - Tu_j\|_{L^p(\partial\Omega)} = 0 \,.$$

Since the sequence u_j converges uniformly to u if $u \in C^0(\overline{\Omega})$, we see that $Tu = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cup C^0(\overline{\Omega})$. □

Sketch of the proof of Theorem 2.32. Just as in the proof of the trace theorem, first suppose that $u \in C^1(\overline{\omega})$ and that near $z \in \partial\Omega$, $\partial\Omega$ is *locally flat*, so that for some $r > 0$, $\partial\Omega \cup B(z, r) \subset \{x_n = 0\}$. Letting $B^+ = B(z, r) \cup \{x_n \geq 0\}$ and $B^- = B(z, r) \cup \{x_n \leq 0\}$, we define the extension of u by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, ..., x_{n-1}, -x_n) + 4u(x_1, ..., x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^- \,. \end{cases}$$

Define $u^+ = \bar{u}|_{B^+}$ and $u^- = \bar{u}|_{B^-}$.

It is clear that $u^+ = u^-$ on $\{x_n = 0\}$, and by the chain-rule, it follows that

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u^-}{\partial x_n}(x_1, ..., -x_n) - 2\frac{\partial u^-}{\partial x_n}(x_1, ..., -\frac{x_n}{2}) \,,$$

so that $\frac{\partial u^+}{\partial x_n} = \frac{\partial u^-}{\partial x_n}$ on $\{x_n = 0\}$. This shows that $\bar{u} \in C^1(B(z, r)$. using the charts θ_l to locally straighten the boundary, and the density of the $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$, the theorem is proved.

□

2.10. The subspace $W_0^{1,p}(\Omega)$.

Definition 2.34. *We let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.*

Theorem 2.35. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded with C^1 boundary, and that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \quad \textit{iff } Tu = 0 \quad \textit{on} \quad \partial\Omega \,.$$

We can now state the Sobolev embedding theorems for bounded domains Ω.

Theorem 2.36 (Gagliardo-Nirenberg inequality for $W^{1,p}(\Omega)$)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary, $1 \le p < n$, and $u \in W^{1,p}(\Omega)$. Then*

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \text{ for a constant } C = C(p,n,\Omega)\,.$$

Proof. Choose $\tilde{\Omega} \subset \mathbb{R}^n$ bounded such that $\Omega \subset\subset \tilde{\Omega}$, and let Eu denote the Sobolev extension of u to $\mathbb{R}^n$ such that $Eu = u$ a.e., $\operatorname{spt}(Eu) \subset \tilde{\Omega}$, and $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$.

Then by the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \le \|Eu\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \le C\|D(Eu)\|_{L^p(\mathbb{R}^n)} \le C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}\,.$$

□

Theorem 2.37 (Gagliardo-Nirenberg inequality for $W_0^{1,p}(\Omega)$)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary, $1 \le p < n$, and $u \in W_0^{1,p}(\Omega)$. Then for all $1 \le q \le \frac{np}{n-p}$,*

$$\|u\|_{L^q(\Omega)} \le C\|Du\|_{L^p(\Omega)} \text{ for a constant } C = C(p,n,\Omega)\,. \tag{2.12}$$

Proof. By definition there exists a sequence $u_j \in C_0^\infty(\Omega)$ such that $u_j \to u$ in $W^{1,p}(\Omega)$. Extend each u_j by 0 on Ω^c. Applying Theorem 2.29 to this extension, and using the continuity of the norms, we obtain $\|u\|_{L^{\frac{pn}{n-p}}(\Omega)} \le C\|Du\|_{L^p(\Omega)}$. Since Ω is bounded, the assertion follows by Hölder's inequality. □

Theorem 2.38. *Suppose that $\Omega \subset \mathbb{R}^2$ is open and bounded with C^1 boundary, and $u \in H_0^1(\Omega)$. Then for all $1 \le q < \infty$,*

$$\|u\|_{L^q(\Omega)} \le C\sqrt{q}\|Du\|_{L^2(\Omega)} \text{ for a constant } C = C(\Omega)\,. \tag{2.13}$$

Proof. The proof follows that of Theorem 2.30. Instead of introducing the cut-off function g, we employ a partition of unity subordinate to the finite covering of the bounded domain Ω, in which case it suffices that assume that $\operatorname{spt}(u) \subset \operatorname{spt}(U)$ with U also defined in the proof Theorem 2.30. □

Remark 2.39. *Inequalities (2.12) and (2.13) are commonly referred to as Poincaré inequalities. They are invaluable in the study of the Dirichlet problem for Poisson's equation, since the right-hand side provides an $H^1(\Omega)$-equivalent norm for all $u \in H_0^1(\Omega)$. In particular, there exists constants C_1, C_2 such that*

$$C_1\|Du\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)} \le C_2\|Du\|_{L^2(\Omega)}\,.$$

2.11. **Weak solutions to Dirichlet's problem.** Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with C^1 boundary. A classical problem in the linear theory of partial differential equations consists of finding solutions to the *Dirichlet problem*:

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \tag{2.14a}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{2.14b}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator or *Laplacian*. As written, (2.14) is the so-called *strong form* of the Dirichlet problem, as it requires that u to possess certain weak second-order partial derivatives. A major turning-point in the modern theory of linear partial differential equations was the realization that *weak* *solutions* of (2.14) could be defined, which only require weak first-order derivatives of u to exist. (We will see more of this idea later when we discuss the theory of distributions.)

Definition 2.40. *The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$. For $f \in H^{-1}(\Omega)$,*

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\|\psi\|_{H^1_0(\Omega)}=1} \langle f, \psi \rangle ,$$

where $\langle f, \psi\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Definition 2.41. *A function $u \in H^1_0(\Omega)$ is a weak solution of (2.14) if*

$$\int_\Omega Du \cdot Dv \, dx = \langle f, v\rangle \quad \forall v \in H^1_0(\Omega) .$$

Remark 2.42. *Note that f can be taken in $H^{-1}(\Omega)$. According to the Sobolev embedding theorem, this implies that when $n = 1$, the forcing function f can be taken to be the Dirac Delta distribution.*

Remark 2.43. *The motivation for Definition 2.41 is as follows. Since $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$, multiply equation (2.14a) by $\phi \in C^\infty_0(\Omega)$, integrate over Ω, and employ the integration-by-parts formula to obtain $\int_\Omega Du \cdot D\phi \, dx = \int_\Omega f\phi \, dx$; the boundary terms vanish because ϕ is compactly supported.*

Theorem 2.44 (Existence and uniqueness of weak solutions)**.** *For any $f \in H^{-1}(\Omega)$, there exists a unique weak solution to (2.14).*

Proof. Using the Poincaré inequality, $\|Du\|_{L^2(\Omega)}$ is an H^1-equivalent norm for all $u \in H^1_0(\Omega)$, and $(Du, Dv)_{L^2(\Omega)}$ defines the inner-product on $H^1_0(\Omega)$. As such, according to the definition of weak solutions to (2.14), we are seeking $u \in H^1_0(\Omega)$ such that

$$(u, v)_{H^1_0(\Omega)} = \langle f, v\rangle \quad \forall v \in H^1_0(\Omega) . \tag{2.15}$$

The existence of a unique $u \in H^1_0(\Omega)$ satisfying (2.15) is provided by the Riesz representation theorem for Hilbert spaces. □

Remark 2.45. *Note that the Riesz representation theorem shows that there exists a distribution, denote $-\Delta u \in H^{-1}(\Omega)$ such that*

$$\langle -\Delta u, v\rangle = \langle f, v\rangle \quad \forall v \in H^1_0(\Omega) .$$

The operator $-\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is thus an isomorphism.

A fundamental question in the theory of linear partial differential equations is commonly referred to as *elliptic regularity*, and can be explained as follows: in order to develop an existence and uniqueness theorem for the Dirichlet problem, we have significantly generalized the notion of solution to the class of weak solutions, which permitted very weak forcing functions in $H^{-1}(\Omega)$. Now suppose that the forcing function is smooth; is the weak solution smooth as well? Furthermore, does the weak solution agree with the classical solution? The answer is yes, and we will develop this regularity theory in Chapter 6, where it will be shown that for integers $k \geq 2$, $-\Delta : H^k(\Omega) \cap H^1_0(\Omega) \to H^{k-2}(\Omega)$ is also an isomorphism. An important consequence of this result is that $(-\Delta)^{-1} : H^{k-2}(\Omega) \to H^k(\Omega) \cap H^1_0(\Omega)$ is a *compact* linear operator, and as such has a countable set of eigenvalues, a fact that is eminently useful in the construction of solutions for heat- and wave-type equations.

For this reason, as well as the consideration of weak limits of nonlinear combinations of sequences, we must develop a compactness theorem, which generalizes the well-known Arzela-Ascoli theorem to Sobolev spaces.

2.12. **Strong compactness.** In Section 1.11, we defined the notion of weak converence and weak compactness for L^p-spaces. Recall that for $1 \le p < \infty$, a sequence $u_j \in L^p(\Omega)$ converges *weakly* to $u \in L^p(\Omega)$, denoted $u_j \rightharpoonup u$ in $L^p(\Omega)$, if $\int_\Omega u_j v dx \to \int_\Omega uv dx$ for all $v \in L^q(\Omega)$, with $q = \frac{p}{p-1}$. We can extend this definition to Sobolev spaces.

Definition 2.46. *For $1 \le p < \infty$, $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ provided that $u_j \rightharpoonup u$ in $L^p(\Omega)$ and $Du_j \rightharpoonup Du$ in $L^p(\Omega)$.*

Alaoglu's Lemma (Theorem 1.37) then implies the following theorem.

Theorem 2.47 (Weak compactness in $W^{1,p}(\Omega)$)**.** *For $\Omega \subset \mathbb{R}^n$, suppose that*

$$\sup \|u_j\|_{W^{1,p}(\Omega)} \le M < \infty \quad \text{for a constant } M \ne M(j)\,.$$

Then there exists a subsequence $u_{j_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$.

It turns out that weak compactness often does not suffice for limit processes involving nonlinearities, and that the Gagliardo-Nirenberg inequality can be used to obtain the following strong compactness theorem.

Theorem 2.48 (Rellich's theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with C^1 boundary, and that $1 \le p < n$. Then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \le q < \frac{np}{n-p}$, i.e. if*

$$\sup \|u_j\|_{W^{1,p}(\Omega)} \le M < \infty \quad \text{for a constant } M \ne M(j)\,,$$

then there exists a subsequence $u_{j_k} \to u$ in $L^q(\Omega)$. In the case that $n = 2$ and $p = 2$, $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $1 \le q < \infty$.

In order to prove Rellich's theorem, we need two lemmas.

Lemma 2.49 (Arzela-Ascoli Theorem)**.** *Suppose that $u_j \in C^0(\overline{\Omega})$, $\|u_j\|_{C^0(\overline{\Omega})} \le M < \infty$, and u_j is equicontinuous. Then there exists a subsequence $u_{j_k} \to u$ uniformly on $\overline{\Omega}$.*

Lemma 2.50. *Let $1 \le r \le s \le t \le \infty$, and suppose that $u \in L^r(\Omega) \cap L^t(\Omega)$. Then for $\frac{1}{s} = \frac{a}{r} + \frac{1-a}{t}$*

$$\|u\|_{L^s(\Omega)} \le \|u\|^a_{L^r(\Omega)} \|u\|^{1-a}_{L^t(\Omega)}\,.$$

Proof. By Hölder's inequality,

$$\begin{aligned}\int_\Omega |u|^s dx &= \int_\Omega |u|^{as} |u|^{(1-a)s} dx \\ &\le \left(\int_\Omega |u|^{as\frac{r}{as}} dx\right)^{\frac{as}{r}} \left(\int_\Omega |u|^{(1-a)s\frac{t}{(1-a)s}} dx\right)^{\frac{(1-a)s}{t}} = \|u\|^{as}_{L^r(\Omega)} \|u\|^{(1-a)s}_{L^t(\Omega)}\,.\end{aligned}$$

□

Proof of Rellich's theorem. Let $\tilde{\Omega} \subset \mathbb{R}^n$ denote an open, bounded domain such that $\Omega \subset\subset \tilde{\Omega}$. By the Sobolev extension theorem, the sequence u_j satisfies $\operatorname{spt}(u_j) \subset \tilde{\Omega}$, and

$$\sup \|Eu_j\|_{W^{1,p}(\mathbb{R}^n)} \leq CM\,.$$

Denote the sequence Eu_j by $\bar{u}_j$. By the Gagliardo-Nirenberg inequality, if $1 \leq q < \frac{np}{n-p}$,

$$\sup \|u\|_{L^q(\Omega)} \leq \sup \|\bar{u}\|_{L^q(\mathbb{R}^n)} \leq C \sup \|\bar{u}_j\|_{W^{1,p}(\mathbb{R}^n)} \leq CM\,.$$

For $\epsilon > 0$, let η_ϵ denote the standard mollifiers and set $\bar{u}_j^\epsilon = \eta_\epsilon * Eu_j$. By choosing $\epsilon > 0$ sufficiently small, $\bar{u}_j^\epsilon \in C_0^\infty(\tilde{\Omega})$. Since

$$\bar{u}_j^\epsilon = \int_{B(0,\epsilon)} \frac{1}{\epsilon^n} \eta(\frac{y}{\epsilon}) \bar{u}_j(x-y) dy = \int_{B(0,1)} \eta(z) \bar{u}_j(x - \epsilon z) dz\,,$$

and if $\bar{u}_j$ is smooth,

$$\bar{u}_j(x-\epsilon z) - \bar{u}_j(x) = \int_0^1 \frac{d}{dt} \bar{u}_j(x - \epsilon t z) dt = -\epsilon \int_0^1 D\bar{u}_j(x-\epsilon t z) \cdot z\, dt\,.$$

Hence,

$$|\bar{u}_j^\epsilon(x) - \bar{u}_j(x)| = \epsilon \int_{B(0,1)} \eta(z) \int_0^1 |D\bar{u}_j(x - \epsilon t z)|\, dz dt\,,$$

so that

$$\begin{aligned}\int_{\tilde{\Omega}} |\bar{u}_j^\epsilon(x) - \bar{u}_j(x)| dx &= \epsilon \int_{B(0,1)} \eta(z) \int_0^1 \int_{\tilde{\Omega}} |D\bar{u}_j(x-\epsilon t z)|\, dx dz dt \\ &\leq \epsilon \|D\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq \epsilon \|D\bar{u}_j\|_{L^p(\tilde{\Omega})} < \epsilon CM\,.\end{aligned}$$

Using the L^p-interpolation Lemma 2.50,

$$\begin{aligned}\|\bar{u}_j^\epsilon - \bar{u}_j\|_{L^q(\tilde{\Omega})} &\leq \|\bar{u}_j^\epsilon - \bar{u}_j\|^a_{L^1(\tilde{\Omega})} \|\bar{u}_j^\epsilon - \bar{u}_j\|^{1-a}_{L^{\frac{np}{n-p}}(\tilde{\Omega})} \\ &\leq \epsilon CM \, \|D\bar{u}_j^\epsilon - D\bar{u}_j\|^{1-a}_{L^p(\tilde{\Omega})} \\ &\leq \epsilon CM M^{1-a}\end{aligned} \tag{2.16}$$

The inequality (2.16) shows that $\bar{u}_j^\epsilon$ is arbitrarily close to $\bar{u}_j$ in $L^q(\Omega)$ uniformly in $j \in \mathbb{N}$; as such, we attempt to use the smooth sequence $\bar{u}_j^\epsilon$ to construct a convergent subsequence $\bar{u}_{j_k}^\epsilon$. Our goal is to employ the Arzela-Ascoli Theorem, so we show that for $\epsilon > 0$ fixed,

$$\|\bar{u}_j^\epsilon\|_{C^0(\tilde{\Omega})} \leq \tilde{M} < \infty \quad \text{and } \bar{u}_j^\epsilon \text{ is equicontinous.}$$

For $x \in \mathbb{R}^n$,

$$\begin{aligned}\sup_j \|\bar{u}_j^\epsilon\|_{C^0(\overline{\Omega})} &\leq \sup_j \int_{B(x,\epsilon)} \eta_\epsilon(x-y) |\bar{u}_j(y)| dy \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \sup_j \|\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq C\epsilon^{-n} < \infty\,,\end{aligned}$$

and similarly

$$\sup_j \|\bar{D}u_j^\epsilon\|_{C^0(\overline{\Omega})} \leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \sup_j \|\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq C\epsilon^{-n-1} < \infty\,.$$

The latter inequality proves equicontinuity of the sequence $\bar{u}_j^\epsilon$, and hence there exists a subsequence u_{j_k} which converges uniformly on $\tilde{\Omega}$, so that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k}^\epsilon - \bar{u}_{j_l}^\epsilon\|_{L^q(\tilde{\Omega})} = 0\,.$$

It follows from (2.16) and the triangle inequality that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k} - \bar{u}_{j_l}\|_{L^q(\tilde{\Omega})} \leq C\epsilon\,.$$

Letting $C\epsilon = 1, \frac{1}{2}, \frac{1}{3}$, etc., and using the diagonal argument to extract further subsequences, we can arrange to find a subsequence again denoted by $\{\bar{u}_{j_k}\}$ of $\{\bar{u}_j\}$ such that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k} - \bar{u}_{j_l}\|_{L^q(\tilde{\Omega})} = 0\,,$$

and hence

$$\limsup_{k,l\to\infty} \|u_{j_k} - u_{j_l}\|_{L^q(\Omega)} = 0\,,$$

The case that $n = p = 2$ follows from Theorem 2.30. □

3. The Fourier Transform

3.1. Fourier transform on $L^1(\mathbb{R}^n)$ and the space $\mathcal{S}(\mathbb{R}^n)$.

Definition 3.1. *For all $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\mathcal{F}$ is defined by*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx\,.$$

By Hölder's inequality, $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

Definition 3.2. *The space of Schwartz functions of rapid decay is denoted by*

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n) \quad \forall \alpha, \beta \in \mathbb{Z}^n_+\}.$$

It is not difficult to show that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,$$

and that

$$\xi^\alpha D_\xi^\beta \hat{f} = (-i)^{|\alpha|}(-1)^{|\beta|}\mathcal{F}(D_x^\alpha x^\beta f)\,.$$

Definition 3.3. *For all $f \in L^1(\mathbb{R}^n)$, we define operator $\mathcal{F}^*$ by*

$$\mathcal{F}^* f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi\,.$$

Lemma 3.4. *For all $u, v \in \mathcal{S}(\mathbb{R}^n)$,*

$$(\mathcal{F}u, v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^* v)_{L^2(\mathbb{R}^n)}\,.$$

Recall that the $L^2(\mathbb{R}^n)$ inner-product for complex-valued functions is given by $(u,v)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\overline{v(x)} dx$.

Proof. Since $u, v \in \mathcal{S}(\mathbb{R}^n)$, by Fubini's Theorem,

$$
\begin{aligned}
(\mathcal{F}u, v)_{L^2(\mathbb{R}^n)} &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx\, \overline{v(\xi)}\, d\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) \overline{e^{ix\cdot\xi} v(\xi)}\, d\xi\, dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \overline{e^{ix\cdot\xi} v(\xi)}\, d\xi\, dx = (u, \mathcal{F}^* v)_{L^2(\mathbb{R}^n)}\,,
\end{aligned}
$$

□

Theorem 3.5. *$\mathcal{F}^* \circ \mathcal{F} = \mathrm{Id} = \mathcal{F} \circ \mathcal{F}^*$ on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. We first prove that for all $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}^*\mathcal{F}f(x) = f(x)$.

$$
\begin{aligned}
\mathcal{F}^*\mathcal{F}f(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y) dy \right) d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi\,.
\end{aligned}
$$

By the dominated convergence theorem,

$$
\mathcal{F}^*\mathcal{F}f(x) = \lim_{\epsilon\to 0} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi\,.
$$

For all $\epsilon > 0$, the *convergence factor* $e^{-\epsilon|\xi|^2}$ allows us to interchange the order of integration, so that by Fubini's theorem,

$$
\mathcal{F}^*\mathcal{F}f(x) = \lim_{\epsilon\to 0} (2\pi)^{-n} \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} e^{i(y-x)\cdot\xi}\, d\xi \right) dy\,.
$$

Define the integral kernel

$$
p_\epsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2 + ix\cdot\xi} d\xi
$$

Then

$$
\mathcal{F}^*\mathcal{F}f(x) = \lim_{\epsilon\to 0} p_\epsilon * f := \int_{\mathbb{R}^n} p_\epsilon(x-y) f(y) dy\,.
$$

Let $p(x) = p_1(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^2 + ix\cdot\xi} d\xi$. Then

$$
\begin{aligned}
p(x/\sqrt{\epsilon}) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^2 + ix\cdot\xi/\sqrt{\epsilon}} d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^2 + ix\cdot\xi} \epsilon^{\frac{n}{2}} d\xi = \epsilon^{\frac{n}{2}} p_\epsilon(x)\,.
\end{aligned}
$$

We claim that

$$
p_\epsilon(x) = \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\epsilon}} \quad \text{and that} \quad \int_{\mathbb{R}^n} p(x) dx = 1\,. \tag{3.1}
$$

Given (3.1), then for all $f \in \mathcal{S}(\mathbb{R}^n)$, $p_\epsilon * f \to f$ uniformly as $\epsilon \to 0$, which shows that $\mathcal{F}^*\mathcal{F} = \mathrm{Id}$, and similar argument shows that $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$. (Note that this follows from the proof of Theorem 1.28, since the standard mollifiers η_ϵ can be replaced by the sequence p_ϵ and all assertions of the theorem continue to hold, for if (3.1) is

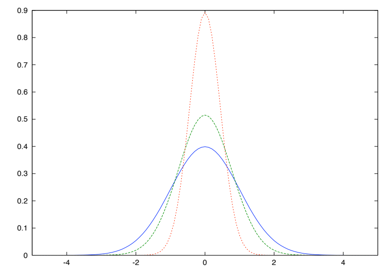

FIGURE 1. As $\epsilon \to 0$, the sequence of functions p_ϵ becomes more localized about the origin.

true, then even though p_ϵ does not have compact support, $\int_{B(0,\delta)^c} p_\epsilon(x)dx \to 0$ as $\epsilon \to 0$ for all $\delta > 0$.)

Thus, it remains to prove (3.1). It suffices to consider the case $\epsilon = \frac{1}{2}$; then by definition

$$p_{\frac{1}{2}}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|^2}{2}} d\xi$$
$$= \mathcal{F}\left((2\pi)^{-n/2} e^{-\frac{|\xi|^2}{2}} \right) .$$

In order to prove that $p_{\frac{1}{2}}(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$, we must show that with the Gaussian function $G(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$,

$$G(x) = \mathcal{F}(G(\xi)) .$$

By the multiplicative property of the exponential,

$$e^{-|\xi|^2/2} = e^{-\xi_1^2/2} \cdots e^{-\xi_n^2/2} ,$$

it suffices to consider the case that $n = 1$. Then the Gaussian satisfies the differential equation

$$\frac{d}{dx} G(x) + xG(x) = 0 .$$

Computing the Fourier transform, we see that

$$-i \frac{d}{d\xi} \hat{G}(x) - i\xi \hat{G}(x) = 0 .$$

Thus,

$$\hat{G}(\xi) = Ce^{-\frac{\xi^2}{2}} .$$

To compute the constant C,

$$C = \hat{G}(0) = (2\pi)^{-1} \int_{\mathbb{R}} e^{\frac{x^2}{2}} dx = (2\pi)^{-\frac{1}{2}}$$

which follows from the fact that

$$\int_{\mathbb{R}} e^{\frac{x^2}{2}} dx = (2\pi)^{\frac{1}{2}} . \tag{3.2}$$

To prove (3.2), one can again rely on the multiplication property of the exponential to observe that

$$\int_{\mathbb{R}} e^{\frac{x_1^2}{2}} dx \int_{\mathbb{R}} e^{\frac{x_2^2}{2}} dx = \int_{\mathbb{R}^2} e^{\frac{x_1^2+x_2^2}{2}} dx$$
$$= \int_0^{2\pi} \int_0^\infty e^{-2r^2} r dr d\theta = 2\pi \,.$$

□

It follows from Lemma 3.4 that for all $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^*\mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} \,.$$

Thus, we have established the *Plancheral theorem* on $\mathcal{S}(\mathbb{R}^n)$.

Theorem 3.6 (Plancheral's theorem)**.** *$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse $\mathcal{F}^*$ preserving the $L^2(\mathbb{R}^n)$ inner-product.*

3.2. **The topology on $\mathcal{S}(\mathbb{R}^n)$ and tempered distributions.** An alternative to Definition 3.2 can be stated as follows:

Definition 3.7 (The space $\mathcal{S}(\mathbb{R}^n)$)**.** *Setting $\langle x \rangle = \sqrt{1+|x|^2}$,*

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid \langle x \rangle^k |D^\alpha u| \le C_{k,\alpha} \quad \forall k \in \mathbb{Z}_+\} \,.$$

The space $\mathcal{S}(\mathbb{R}^n)$ has a Fréchet topology determined by seminorms.

Definition 3.8 (Topology on $\mathcal{S}(\mathbb{R}^n)$)**.** *For $k \in \mathbb{Z}_+$, define the semi-norm*

$$p_k(u) = \sup_{x \in \mathbb{R}^n, |\alpha| \le k} \langle x \rangle^k |D^\alpha u(x)| \,,$$

and the metric on $\mathcal{S}(\mathbb{R}^n)$

$$d(u, v) = \sum_{k=0}^\infty 2^{-k} \frac{p_k(u-v)}{1+p_k(u-v)} \,.$$

The space $(\mathcal{S}(\mathbb{R}^n), d)$ is a Fréchet space.

Definition 3.9 (Convergence in $\mathcal{S}(\mathbb{R}^n)$)**.** *A sequence $u_j \to u$ in $\mathcal{S}(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ as $j \to \infty$ for all $k \in \mathbb{Z}_+$.*

Definition 3.10 (Tempered Distributions)**.** *A linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if there exists some $k \in \mathbb{Z}_+$ and constant C such that*

$$|\langle T, u \rangle| \le C p_k(u) \quad \forall u \in \mathcal{S}(\mathbb{R}^n) \,.$$

The space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Elements of $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions.

Definition 3.11 (Convergence in $\mathcal{S}'(\mathbb{R}^n)$)**.** *A sequence $T_j \rightharpoonup T$ in $\mathcal{S}'(\mathbb{R}^n)$ if $\langle T_j, u \rangle \to \langle T, u \rangle$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.*

For $1 \le p \le \infty$, there is a natural injection of $L^p(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle f, u \rangle = \int_{\mathbb{R}^n} f(x) u(x) dx \quad \forall u \in \mathcal{S}(\mathbb{R}^n) \,.$$

Any finite measure on $\mathbb{R}^n$ provides an element of $\mathcal{S}'(\mathbb{R}^n)$. The basic example of such a finite measure is the Dirac delta 'function' defined as follows:

$$\langle \delta, u \rangle = u(0) \quad \text{or, more generally,} \quad \langle \delta_x, u \rangle = u(x) \quad \forall u \in \mathcal{S}(\mathbb{R}^n) \,.$$

Definition 3.12. *The distributional derivative* $D : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is defined by the relation*

$$\langle DT, u\rangle = -\langle T, Du\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

More generally, the α*th distributional derivative exists in* $\mathcal{S}'(\mathbb{R}^n)$ *and is defined by*

$$\langle D^\alpha T, u\rangle = (-1)^{|\alpha|}\langle T, D^\alpha u\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Multiplication by $f \in \mathcal{S}(\mathbb{R}^n)$ preserves $\mathcal{S}'(\mathbb{R}^n)$; in particular, if $T \in \mathcal{S}'(\mathbb{R}^n)$, then $fT \in \mathcal{S}(\mathbb{R}^n)$ and is defined by

$$\langle fT, u\rangle = \langle T, fu\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Example 3.13. *Let* $H := \mathbf{1}_{[0,\infty)}$ *denote the Heavyside function. Then*

$$\frac{dH}{dx} = \delta \quad in \quad \mathcal{S}'(\mathbb{R}^n)\,.$$

This follows since for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \frac{dH}{dx}, u\rangle = -\langle H, \frac{du}{dx}\rangle = -\int_0^\infty \frac{du}{dx}dx = u(0) = \langle \delta, u\rangle\,.$$

Example 3.14 (Distributional derivative of Dirac measure)**.**

$$\langle \frac{d\delta}{dx}, u\rangle = -\frac{du}{dx}(0) \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

3.3. **Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$.**

Definition 3.15. *Define* $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *by*

$$\langle \mathcal{F}T, u\rangle = \langle T, \mathcal{F}u\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,,$$

with the analogous definition for $\mathcal{F}^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Theorem 3.16. $\mathcal{F}\mathcal{F}^* = \mathrm{Id} = \mathcal{F}^*\mathcal{F}$ *on* $\mathcal{S}'(\mathbb{R}^n)\,.$

Proof. By Definition 3.15, for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}\mathcal{F}^*T, u\rangle = \langle \mathcal{F}^*w, \mathcal{F}u\rangle = \langle T, \mathcal{F}^*\mathcal{F}u\rangle = \langle T, u\rangle\,,$$

the last equality following from Theorem 3.5. □

Example 3.17 (Fourier transform of δ)**.** *We claim that* $\mathcal{F}\delta = (2\pi)^{-\frac{n}{2}}$. *According to Definition 3.15, for all* $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \mathcal{F}\delta, u\rangle = \langle \delta, \mathcal{F}u\rangle = \mathcal{F}u(0) = \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}}\, u(x)dx\,,$$

so that $\mathcal{F}\delta = (2\pi)^{-\frac{n}{2}}$.

Example 3.18. *The same argument shows that* $\mathcal{F}^*\delta = (2\pi)^{-\frac{n}{2}}$ *so that* $\mathcal{F}^*[(2\pi)^{\frac{n}{2}}] = 1$. *Using Theorem 3.16, we see that* $\mathcal{F}(1) = (2\pi)^{-\frac{n}{2}}\delta$. *This demonstrates nicely the identity*

$$|\xi^\alpha \hat{u}(\xi)| = |D^\alpha u(x)|.$$

In other the words, the smoother the function $x \mapsto u(x)$ *is, the faster* $\xi \mapsto \hat{u}(\xi)$ *must decay.*

3.4. **The Fourier transform on $L^2(\mathbb{R}^n)$.** In Theorem 1.28, we proved that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. Since $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ as well. Thus, for every $u \in L^2(\mathbb{R}^n)$, there exists a sequence $u_j \in \mathcal{S}(\mathbb{R}^n)$ such that $u_j \to u$ in $L^2(\mathbb{R}^n)$, so that by Plancheral's Theorem 3.6,

$$\|\hat{u}_j - \hat{u}_k\|_{L^2(\mathbb{R}^n)} = \|u_j - u_k\|_{L^2(\mathbb{R}^n)} < \epsilon\,.$$

It follows from the completeness of $L^2(\mathbb{R}^n)$ that the sequence $\hat{u}_j$ *converges* in $L^2(\mathbb{R}^n)$.

Definition 3.19 (Fourier transform on $L^2(\mathbb{R}^n)$)**.** *For $u \in L^2(\mathbb{R}^n)$ let u_j denote an approximating sequence in $\mathcal{S}(\mathbb{R}^n)$. Define the Fourier transform as follows:*

$$\mathcal{F}u = \hat{u} = \lim_{j\to\infty} \hat{u}_j\,.$$

Note well that $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ is well-defined, as the limit is independent of the approximating sequence. In particular,

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} \|\hat{u}_j\|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} \|u_j\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}\,.$$

By the polarization identity

$$(u,v)_{L^2(\mathbb{R}^n)} = \frac{1}{2}\Big(\|u+v\|^2_{L^2(\mathbb{R}^n)} - i\|u+iv\|^2_{L^2(\mathbb{R}^n)} - (1-i)\|u\|^2_{L^2(\mathbb{R}^n)} - (1-i)\|v\|^2_{L^2(\mathbb{R}^n)}\Big)$$

we have proved the Plancheral theorem[1] on $L^2(\mathbb{R}^n)$:

Theorem 3.20. $(u,v)_{L^2(\mathbb{R}^n)} = (\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} \quad \forall u, v \in L^2(\mathbb{R}^n)\,.$

3.5. **Bounds for the Fourier transform on $L^p(\mathbb{R}^n)$.** We have shown that for $u \in L^1(\mathbb{R}^n)$, $\|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}}\|u\|_{L^1(\mathbb{R}^n)}$, and that for $u \in L^2(\mathbb{R}^n)$, $\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$. Interpolating p between 1 and 2 yields the following result.

Theorem 3.21 (Hausdorff-Young inequality)**.** *If $u \in L^p(\mathbb{R}^n)$ for $1 \le p \le 2$, the for $q = \frac{p-1}{p}$, there exists a constant C such that*

$$\|\hat{u}\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}\,.$$

Returning to the case that $u \in L^1(\mathbb{R}^n)$, not only is $\mathcal{F}u \in L^\infty(\mathbb{R}^n)$, but the transformed function decays at infinity.

Theorem 3.22 (Riemann-Lebesgue "lemma")**.** *For $u \in L^1(\mathbb{R}^n)$, $\mathcal{F}u$ is continuous and $\mathcal{F}u(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof. Let $B_M = B(0,M) \subset \mathbb{R}^n$. Since $f \in L^1(\mathbb{R}^n)$, for each $\epsilon > 0$, we can choose M sufficiently large such that $\hat{f}(\xi) \le \epsilon + \int_{B_M} e^{-ix\cdot\xi}|f(x)|dx$. Using Lemma 1.23, choose a sequence of simple functions $\phi_j(x) \to f(x)$ a.e. on B_M. For $jn\mathbb{N}$ chosen sufficiently large,

$$\hat{f}(\xi) \le 2\epsilon + \int_{B_M} \phi_j(x)e^{-ix\cdot\xi}dx\,.$$

Write $\phi_j(x) = \sum_{l=1}^N C_l \mathbf{1}_{E_l}(x)$ so that

$$\hat{f}(\xi) \le 2\epsilon + \sum_{l=1}^N C_l \int_{E_l} \phi_j(x)e^{-ix\cdot\xi}dx\,.$$

[1]The unitarity of the Fourier transform is often called Parseval's theorem in science and engineering fields, based on an earlier (but less general) result that was used to prove the unitarity of the Fourier series.

By the regularity of the Lebesgue measure μ, for all $\epsilon > 0$ and each $l \in \{1, ..., N\}$, there exists a compact set K_l and an open set O_l such that

$$\mu(O_l) - \epsilon/2 < \mu(E_l) < \mu(K_l) + \epsilon/2\,.$$

Then $O_l = \{\cup_{\alpha\in A_l} V^l_\alpha \mid V^\alpha_l \subset \mathbb{R}^n$ is open rectangle , A_l arbitrary set $\}$, and $K_l \subset \cup_{j=1}^{N_l} V_j^l \subset O_l$ where $\{1, ..., N_l\} \subset A_l$ such that

$$|\mu(E_l) - \mu(\cup_{j=1}^{N_l} V_j^l)| < \epsilon\,.$$

It follows that

$$\left| \int_{E_l} e^{-ix\cdot\xi} dx - \int_{\cup_{j=1}^{N_l} V_j^l} e^{-ix\cdot\xi} dx \right| < \epsilon\,.$$

On the other hand, for each rectangle V_j^l, $\int_{V_j^l} e^{-ix\cdot\xi} dx| \leq C/(\xi_1 \cdots \xi_n)$, so that

$$\hat{f}(\xi) \leq C \left(\epsilon + \frac{1}{\xi_1 \cdots \xi_n} \right)\,.$$

Since $\epsilon > 0$ is arbitrary, we see that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. Continuity of $\mathcal{F}u$ follows easily from the dominated convergence theorem. □

3.6. **The Fourier transform and convolution.**

Theorem 3.23. *If $u, v \in L^1(\mathbb{R}^n)$, then $u * v \in L^1(\mathbb{R}^n)$ and*

$$\mathcal{F}(u * v) = (2\pi)^{\frac{n}{2}} \mathcal{F}u\, \mathcal{F}v\,.$$

Proof. Young's inequality (Theorem 1.53) shows that $u * v \in L^1(\mathbb{R}^n)$ so that the Fourier transform is well-defined. The assertion then follows from a direct computation:

$$\begin{aligned}
\mathcal{F}(u * v) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (u * v)(x) dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x-y)v(y)dy\, e^{-ix\cdot\xi}\, dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x-y)e^{-i(x-y)\cdot\xi}\, dx\, v(x)\, e^{-iy\cdot\xi}\, dy \\
&= (2\pi)^{\frac{n}{2}}\, \hat{u}\hat{v} \quad \text{(by Fubini's theorem)}\,.
\end{aligned}$$

□

By using Young's inequality (Theorem 1.54) together with the Hausdorff-Young inequality, we can generalize the convolution result to the following

Theorem 3.24. *Suppose that $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, and let r satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ for $1 \leq p, q, r \leq 2$. Then $\mathcal{F}(u * v) \in L^{\frac{r}{r-1}}(\mathbb{R}^n)$ and*

$$\mathcal{F}(u * v) = (2\pi)^{\frac{n}{2}} \mathcal{F}u\, \mathcal{F}v\,.$$

4. The Sobolev Spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$

The Fourier transform allows us to generalize the Hilbert spaces $H^k(\mathbb{R}^n)$ for $k \in \mathbb{Z}_+$ to $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, and hence study functions which possess fractional derivatives (and anti-derivatives) which are square integrable.

Definition 4.1. *For any $s \in \mathbb{R}^n$, let $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, and set*

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \Lambda^s u \in L^2(\mathbb{R}^n)\}, \end{aligned}$$

where $\Lambda^s u = \mathcal{F}^*(\langle\xi\rangle^s \hat{u})$.

The operator Λ^s can be thought of as a "differential operator" of order s, and according to Rellich's theorem, Λ^{-s} is a compact operator, yielding the isomorphism

$$H^s(\mathbb{R}^n) = \Lambda^{-s} L^2(\mathbb{R}^n).$$

Definition 4.2. *The inner-product on $H^s(\mathbb{R}^n)$ is given by*

$$(u,v)_{H^s(\mathbb{R}^n)} = (\Lambda^s u, \Lambda^s v)_{L^2(\mathbb{R}^n)} \quad \forall u,v \in H^s(\mathbb{R}^n).$$

and the norm on $H^s(\mathbb{R}^n)$ is

$$\|u\|^s_{H^s(\mathbb{R}^n)} = (u,u)_{H^s(\mathbb{R}^n)} \quad \forall u \in H^s(\mathbb{R}^n).$$

The completeness of $H^s(\mathbb{R}^n)$ with respect to the $\|\cdot\|_{H^s(\mathbb{R}^n)})$ is induced by the completeness of $L^2(\mathbb{R}^n)$.

Theorem 4.3. *For $s \in \mathbb{R}$, $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n)})$ is a Hilbert space.*

Example 4.4 ($H^1(\mathbb{R}^n)$)**.** *The $H^1(\mathbb{R}^n)$ in Fourier representation is exactly the same as the that given by Definition 2.12:*

$$\begin{aligned} \|u\|^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle\xi\rangle^2 \|\hat{u}(\xi)\|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)\|\hat{u}(\xi)\|^2 d\xi \\ &= \int_{\mathbb{R}^n} (|u(x)|^2 + |Du(x)|^2) dx, \end{aligned}$$

the last equality following from the Plancheral theorem.

Example 4.5 ($H^{\frac{1}{2}}(\mathbb{R}^n)$)**.** *The $H^{\frac{1}{2}}(\mathbb{R}^n)$ can be viewed as interpolating between decay required for $\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u} \in H^1(\mathbb{R}^n)$:*

$$H^{\frac{1}{2}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \sqrt{1+|\xi|^2}|\hat{u}(\xi)|^2\, d\xi < \infty\}.$$

Example 4.6 ($H^{-1}(\mathbb{R}^n)$)**.** *The space $H^{-1}(\mathbb{R}^n)$ can be heuristically described as those distributions whose anti-derivative is in $L^2(\mathbb{R}^n)$; in terms of the Fourier representation, elements of $H^{-1}(\mathbb{R}^n)$ possess a transforms that can grow linearly at infinity:*

$$H^{-1}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|\hat{u}(\xi)|^2}{1+|\xi|^2}\, d\xi < \infty\}.$$

For $T \in H^{-s}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$, the duality pairing is given by

$$\langle T, u\rangle = (\Lambda^{-s}T, \Lambda^s u)_{L^2(\mathbb{R}^n)}\,,$$

from which the following result follows.

Proposition 4.7. *For all $s \in \mathbb{R}$, $[H^s(\mathbb{R}^n)]' = H^{-s}(\mathbb{R}^n)$.*

The ability to define fractional-order Sobolev spaces $H^s(\mathbb{R}^n)$ allows us to refine the estimates of the trace of a function which we previously stated in Theorem 2.33. That result, based on the Gauss-Green theorem, stated that the trace operator was continuous from $H^1(\mathbb{R}^n_+)$ into $L^2(\mathbb{R}^{n-1})$. In fact, the trace operator is continuous from $H^1(\mathbb{R}^n_+)$ into $H^{\frac{1}{2}}(\mathbb{R}^{n-1})$.

To demonstrate the idea, we take $n = 2$. Given a continuous function $u : \mathbb{R}^2 \to \{x_1 = 0\}$, we define the operator

$$Tu = u(0, x_2)\,.$$

The trace theorem asserts that we can extend T to a continuous linear map from $H^1(\mathbb{R}2)$ into $H^{\frac{1}{2}}(\mathbb{R})$ so that we only lose one-half of a derivative.

Theorem 4.8. *$T : H^1(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R})$, and there is a constant C such that*

$$\|Tu\|_{H^{\frac{1}{2}}(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R}^2)}\,.$$

Before we proceed with the proof, we state a very useful result.

Lemma 4.9. *Suppose that $u \in \mathcal{S}(\mathbb{R}^2)$ and define $f(x_2) = u(0, x_2)$. Then*

$$\hat f(\xi_2) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_{\xi_1}} \hat u(\xi_1, \xi_2)d\xi_1\,.$$

Proof. $\hat f(\xi_2) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)d\xi_1$ if and only if $f(\xi_2) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^*\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)d\xi_1$, and

$$\frac{1}{\sqrt{2\pi}}\mathcal{F}^*\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)d\xi_1 = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)d\xi_1 e^{ix_2\xi_2}d\xi_2\,.$$

On the other hand,

$$u(x_1, x_2) = \mathcal{F}^*[\hat u(\xi_1, \xi_2)] = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)e^{ix_1\xi_1 + ix_2\xi_2}d\xi_1 d\xi_2\,,$$

so that

$$u(0, x_2) = \mathcal{F}^*[\hat u(\xi_1, \xi_2)] = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat u(\xi_1, \xi_2)e^{ix_2\xi_2}d\xi_1 d\xi_2\,.$$

□

Proof of Theorem 4.8. Suppose that $u \in \mathcal{S}(\mathbb{R}^2)$ and set $f(x_2) = u(0, x_1)$. According to Lemma 4.9,

$$\begin{aligned}\hat f(\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_{\xi_1}} \hat u(\xi_1, \xi_2)d\xi_1 = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_{\xi_1}} \hat u(\xi_1, \xi_2)\langle\xi\rangle\,\langle\xi\rangle^{-1}d\xi_1 \\ &\le \frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{R}} |\hat u(\xi_1, \xi_2)|^2\langle\xi\rangle^2 d\xi_1\right)^{\frac{1}{2}}\,\left(\int_{\mathbb{R}}\langle\xi\rangle^{-2}d\xi_1\right)^{\frac{1}{2}}\,,\end{aligned}$$

and hence

$$|f(\xi_2)|^2 \le C\int_{\mathbb{R}} |\hat u(\xi_1, \xi_2)|^2\langle\xi\rangle^2 d\xi_1 \int_{\mathbb{R}}\langle\xi\rangle^{-2}d\xi_1\,.$$

The key to this trace estimate is the explicit evaluation of the integral $\int_{\mathbb{R}} \langle\xi\rangle^{-2} d\xi_1$:

$$\int_{\mathbb{R}} \frac{1}{1+\xi_1^2+\xi_2^2} d\xi_1 = \left. \frac{\tan^{-1}\left(\frac{\xi_1}{\sqrt{1+\xi_2^2}}\right)}{\sqrt{1+\xi_2^2}} \right|_{-\infty}^{+\infty} \le \pi(1+\xi_2^2)^{-\frac{1}{2}} \,. \tag{4.1}$$

It follows that $\int_{\mathbb{R}}(1+\xi_2^2)^{-\frac{1}{2}}|\hat{f}(\xi_2)|^2 d\xi_2 \le C \int_{\mathbb{R}} |\hat{u}(\xi_1,\xi_2)|^2 \langle\xi\rangle^2 d\xi_1$, so that integration of this inequality over the set $\{\xi_2 \in \mathbb{R}\}$ yields the result. Using the density of $\mathcal{S}(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ completes the proof. □

The proof of the trace theorem in higher dimensions and for general $H^s(\mathbb{R}^n)$ spaces, $s > \frac{1}{2}$, replacing $H^1(\mathbb{R}^n)$ proceeds in a very similar fashion; the only difference is that the integral $\int_{\mathbb{R}} \langle\xi\rangle^{-2} d\xi_1$ is replaced by $\int_{\mathbb{R}^{n-1}} \langle\xi\rangle^{-2s} d\xi_1 \cdots d\xi_{n-1}$, and instead of obtaining an explicit anti-derivative of this integral, an upper bound is instead found. The result is the following general trace theorem.

Theorem 4.10 (The trace theorem for $H^s(\mathbb{R}^n)$)**.** *For $s > \frac{n}{2}$, the trace operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^n)$ is continuous.*

We can extend this result to open, bounded, C^∞ domains $\Omega \subset \mathbb{R}^n$.

Definition 4.11. *Let $\partial\Omega$ denote a closed C^∞ manifold, and let $\{\omega_l\}_{l=1}^K$ denote an open covering of $\partial\Omega$, such that for each $l \in \{1, 2, ..., K\}$, there exist C^∞-class charts ϑ_l which satisfy*

$$\vartheta_l : B(0, r_l) \subset \mathbb{R}^{n-1} \to \omega_l \ \text{ is a } C^\infty \text{ diffeomorphism}.$$

Next, for each $1 \le l \le K$, let $0 \le \varphi_l \in C_0^\infty(U_l)$ denote a partition of unity so that $\sum_{l=1}^L \varphi_l(x) = 1$ for all $x \in \partial\Omega$. For all real $s \ge 0$, we define

$$H^s(\partial\Omega) = \{u \in L^2(\partial\Omega) \ : \ \|u\|_{H^s(\partial\Omega)} < \infty\}\,,$$

where for all $u \in H^s(\partial\Omega)$,

$$\|u\|^2_{H^s(\partial\Omega)} = \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^s(\mathbb{R}^{n-1})}\,.$$

The space $(H^s(\partial\Omega), \|\cdot\|_{H^s(\partial\Omega)})$ is a Hilbert space by virtue of the completeness of $H^s(\mathbb{R}^{n-1})$; furthermore, any system of charts for $\partial\Omega$ with subordinate partition of unity will produce an equivalent norm.

Theorem 4.12 (The trace map on Ω)**.** *For $s > \frac{n}{2}$, the trace operator $T : \Omega \to \partial\Omega$ is continuous.*

Proof. Let $\{U_l\}_{l=1}^K$ denote an n-dimensional open cover of $\partial\Omega$ such that $U_l \cap \partial\Omega = \omega_l$. Define charts $\theta_l : V_l \to U_l$, as in (2.9) but with each chart being a C^∞ map, such that ϑ_l is equal to the restriction of θ_l to the $(n-1)$-dimensional ball $B(0, r_l) \subset \mathbb{R}^{n-1}$). Also, choose a partition of unity $0 \le \zeta_l \in C_0^\infty(U_l)$ subordinate to the covering U_l such that $\varphi_l = \zeta_l|_{\omega_l}$.

Then by Theorem 4.10, for $s > \frac{1}{2}$,

$$\|u\|^2_{H^{s-\frac{1}{2}}(\partial\Omega)} = \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \le C \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^s(\mathbb{R}^n)} \le C\|u\|^2_{H^s(\Omega)}\,.$$

□

Remark 4.13. *The restriction $s > \frac{n}{2}$ arises from the requirement that*

$$\int_{\mathbb{R}^{n-1}} \langle \xi \rangle^{-2s} d\xi_1 \cdots d\xi_{n-1} < \infty \,.$$

One may then ask if the trace operator T is onto; namely, given $f \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ for $s > \frac{1}{2}$, does there exist a $u \in H^s(\mathbb{R}^n)$ such that $f = Tu$? By essentially reversing the order of the proof of Theorem 4.8, it is possible to answer this question in the affirmative. We first consider the case that $n = 2$ and $s = 1$.

Theorem 4.14. *$T : H^1(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R})$ is a surjection.*

Proof. With $\xi = (\xi_1, \xi_2)$, we define (one of many possible choices) the function u on $\mathbb{R}^2$ via its Fourier representation:

$$\hat{u}(\xi_1, \xi_2) = K \hat{f}(\xi_1) \frac{\langle \xi_1 \rangle}{\langle \xi \rangle^2} \,,$$

for a constant $K \neq 0$ to be determined shortly. To verify that $\|u\|_{H^1(\mathbb{R}^1)} \leq \|f\|_{H^{\frac{1}{2}}(\mathbb{R})}$, note that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 d\xi_1 d\xi_2 &= K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(\xi_1)|^2 \frac{\langle \xi_1 \rangle^2}{\langle \xi \rangle^2} d\xi_1 d\xi_2 \\ &= K \int_{-\infty}^{\infty} |\hat{f}(\xi_1)|^2 (1 + \xi_1^2) \int_{-\infty}^{\infty} \frac{1}{1 + \xi_1^2 + \xi_2^2} d\xi_2 \, d\xi_1 \\ &\leq C \|f\|^2_{H^{\frac{1}{2}}(\mathbb{R})} \,, \end{aligned}$$

where we have used the estimate (4.1) for the inequality above.

It remains to prove that $u(x_1, 0) = f(x_1)$, but by Lemma 4.9, it suffices that

$$\int_{-\infty}^{\infty} \hat{u}(\xi_1, \xi_2) d\xi_2 = \sqrt{2\pi} \hat{f}(\xi_1) \,.$$

Integrating $\hat{u}$, we find that

$$\int_{-\infty}^{\infty} \hat{u}(\xi_1, \xi_2) d\xi_2 = K \hat{f}(\xi_1) \sqrt{1 + \xi_1^2} \int_{-\infty}^{\infty} \frac{1}{1 + \xi_1^2 + \xi_2^2} d\xi_2 \leq K \pi \hat{f}(\xi_1)$$

so setting $K = \sqrt{2\pi}/\pi$ completes the proof. □

A similar construction yields the general result.

Theorem 4.15. *For $s > \frac{1}{2}$, $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is a surjection.*

By using the system of charts employed for the proof of Theorem 4.12, we also have the surjectivity of the trace map on bounded domains.

Theorem 4.16. *For $s > \frac{1}{2}$, $T : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is a surjection.*

The Fourier representation provides a very easy proof of a simple version of the Sobolev embedding theorem.

Theorem 4.17. *For $s > n/2$, if $u \in H^s(\mathbb{R}^n)$, then u is continuous and*

$$\max |u(x)| \leq C \|u\|_{H^s(\mathbb{R}^n)} .$$

Proof. By Theorem 3.6, $u = \mathcal{F}^* \hat{u}$; thus according to Hölder's inequality and the Riemann-Lebesgue lemma (Theorem 3.22), it suffices to show that

$$\|\hat{u}\|_{L^1(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)} .$$

But this follows from the Cauchy-Schwarz inequality since

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{u}(\xi) d\xi &= \int_{\mathbb{R}^n} |\hat{u}(\xi)| \langle\xi\rangle^s \langle\xi\rangle^{-s} d\xi \\ &\le \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-2s} d\xi\right)^{\frac{1}{2}} \\ &\le C\|u\|_{H^s(\mathbb{R}^n)} ,\end{aligned}$$

the latter inequality holding whenever $s > n/2$. □

Example 4.18 (Euler equation on $\mathbb{T}^2$)**.** *On some time interval* $[0, T]$ *suppose that* $u(x, t)$*,* $x \in \mathbb{T}^2, t \in [0, T]$*, is a smooth solution of the Euler equations:*

$$\begin{aligned}\partial_t u + (u \cdot D)u + Dp = 0 \ \text{ in } \mathbb{T}^2 \times (0, T] , \\ \operatorname{div} u = 0 \ \text{ in } \mathbb{T}^2 \times (0, T] ,\end{aligned}$$

with smooth initial condition $u|_{t=0} = u_0$*. Written in components,* $u = (u^1, u^2)$ *satisfies* $u^i_t + u^i,_j\, j^j + p,_i = 0$ *for* $i = 1, 2$*, where we are using the Einstein summation convention for summing repeated indices from* 1 *to* 2 *and where* $u^i,_j = \partial u^i / \partial x_j$ *and* $p,_i = \partial p / \partial x_i$*.*

Computing the $L^2(\mathbb{T}^2)$ *inner-product of the Euler equations with* u *yields the equality*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |u(x, t)|^2 dx + \underbrace{\int_{\mathbb{T}^2} u^i,_j\, u^j u^i dx}_{\mathcal{I}_1} + \underbrace{\int_{\mathbb{T}^2} p,_i\, u^i dx}_{\mathcal{I}_2} = 0 .$$

Notice that

$$\mathcal{I}_1 = \frac{1}{2}\int_{\mathbb{T}^2} (|u|^2),_j\, u^j dx = \frac{1}{2}\int_{\mathbb{T}^2} |u|^2 \operatorname{div} u dx = 0 ,$$

the second equality arising from integration by parts with respect to $\partial/\partial x_j$*. Integration by parts in the integral* $\mathcal{I}_2$ *shows that* $\mathcal{I}_2 = 0$ *as well, from which the conservation law* $\frac{d}{dt}\|u(\cdot, t)\|^2_{L^2(\mathbb{T}^2)}$ *follows.*

To estimate the rate of change of higher-order Sobolev norms of u *relies on the use of the Sobolev embedding theorem. In particular, we claim that on a short enough time interval* $[0, T]$*, we have the inequality*

$$\frac{d}{dt}\|u(\cdot, t)\|^2_{H^3(\mathbb{T}^2)} \le C\|u(\cdot, t)\|^3_{H^3(\mathbb{T}^2)} \tag{4.2}$$

from which it follows that $\|u(\cdot, t)\|^2_{H^3(\mathbb{T}^2)} \le M$ *for some constant* $M < \infty$*.*

To prove (4.2), we compute the $H^3(\mathbb{T}^2)$ *inner-product of the Euler equations with* u*:*

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot, t)\|^2_{H^3(\mathbb{T}^2)} + \sum_{|\alpha| \le 3}\int_{\mathbb{T}^2} D^\alpha u^i,_j\, u^j D^\alpha u^i dx + \sum_{|\alpha| \le 3}\int_{\mathbb{T}^2} D^\alpha p,_i\, D^\alpha u^i dx = 0 .$$

The third integral vanishes by integration by parts and the fact that $D^\alpha \operatorname{div} u = 0$*; thus, we focus on the nonlinearity, and in particular, on the highest-order derivatives* $|\alpha| = 3$*, and use* D^3 *to denote all third-order partial derivatives, as well as* *the notation l.o.t. for lower-order terms. We see that*

$$\int_{\mathbb{T}^2} D^3(u^i,_j u^j) D^3 u^i dx = \underbrace{\int_{\mathbb{T}^2} D^3 u^i,_j \ u^j \ D^3 u^i dx}_{\mathcal{K}_1} + \underbrace{\int_{\mathbb{T}^2} u^i,_j \ D^3 u^j \ D^3 u^i dx}_{\mathcal{K}_2} + \int_{\mathbb{T}^2} \text{l. o. t.}\, dx\,.$$

By definition of being lower-order terms, $\int_{\mathbb{T}^2} \text{l. o. t.}\, dx \le C\|u\|^3_{H^3(\mathbb{T}^2)}$, so it remains to estimate the integrals $\mathcal{K}_1$ and $\mathcal{K}_2$. But the integral $\mathcal{K}_1$ vanishes by the same argument that proved $\mathcal{I}_1 = 0$. On the other hand, the integral $\mathcal{K}_2$ is estimated by Hölder's inequality:

$$|\mathcal{K}_2| \le \|u^i,_j\|_{L^\infty(\mathbb{T}^2)} \|D^3 u^j\|_{H^3(\mathbb{T}^2)} \|D^3 u^i\|_{H^3(\mathbb{T}^2)}\,.$$

Thanks to the Sobolev embedding theorem, for $s = 2$ (s needs only to be greater than 1),

$$\|u^i,_j\|_{L^\infty(\mathbb{T}^2)} \le C\|u^i,_j\|_{H^2(\mathbb{T}^2)} \le \|u\|_{H^3(\mathbb{T}^2)}\,,$$

from which it follows that $\mathcal{K}_2 \le C\|u\|^3_{H^3(\mathbb{T}^2)}$, and this proves the claim.

Note well, that it is the Sobolev embedding theorem that requires the use of the space $H^3(\mathbb{T}^2)$ for this analysis; for example, it would not have been possible to establish the inequality (4.2) with the $H^2(\mathbb{T}^2)$ norm replacing the $H^3(\mathbb{T}^2)$ norm.

5. The Sobolev Spaces $H^s(\mathbb{T}^n)$, $s \in \mathbb{R}$

5.1. **Fourier Series: Revisited.**

Definition 5.1. *For $u \in L^1(\mathbb{T}^n)$, define*

$$\mathcal{F}u(k) = \hat{u}_k = (2\pi)^{-n} \int_{\mathbb{T}_n} e^{-ik\cdot x} u(x) dx\,,$$

and

$$\mathcal{F}^* \hat{u}(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}_k e^{ik\cdot x}\,.$$

Note that $\mathcal{F} : L^1(\mathbb{T}^n) \to l^\infty(\mathbb{Z}^n)$. If u is sufficiently smooth, then integration by parts yields

$$\mathcal{F}(D^\alpha u) = -(-i)^{|\alpha|} k^\alpha \hat{u}_k, \quad k^\alpha = k_1^{\alpha_1} \cdots k_n^{\alpha_n}\,.$$

Example 5.2. *Suppose that $u \in C^1(\mathbb{T}^n)$. Then for $j \in \{1, ..., n\}$,*

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial x_j}\right](k) &= (2\pi)^{-n} \int_{\mathbb{T}^n} \frac{\partial u}{\partial x_j} e^{-ik\cdot x} dx \\ &= -(2\pi)^{-n} \int_{\mathbb{T}^n} u(x)\,(-ik_j)\,e^{-ik\cdot x} dx \\ &= ik_j \hat{u}_k\,.\end{aligned}$$

Note that $\mathbb{T}^n$ is a closed manifold without boundary; alternatively, one may identify $\mathbb{T}^n$ with the $[0,1]^n$ with periodic boundary conditions, i.e., with opposite faces identified.

Definition 5.3. *Let $\mathfrak{s} = \mathcal{S}(\mathbb{Z}^n)$ denote the space of rapidly decreasing functions $\hat{u}$ on $\mathbb{Z}^n$ such that for each $N \in \mathbb{N}$,*

$$p_N(u) = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^N |\hat{u}_k| < \infty\,.$$

Then
$$\mathcal{F} : C^\infty(\mathbb{T}^n) \to \mathfrak{s}\,, \quad \mathcal{F}^* : \mathfrak{s} \to C^\infty(\mathbb{T}^n)\,,$$
and $\mathcal{F}^*\mathcal{F} = \text{Id}$ on $C^\infty(\mathbb{T}^n)$ and $\mathcal{F}\mathcal{F}^* = \text{Id}$ on $\mathfrak{s}$. These properties smoothly extend to the Hilbert space setting:
$$\begin{array}{ll} \mathcal{F} : L^2(\mathbb{T}^n) \to l^2 & \mathcal{F}^* : l^2 \to L^2(\mathbb{T}^n) \\ \mathcal{F}^*\mathcal{F} = \text{Id on } L^2(\mathbb{T}^n) & \mathcal{F}\mathcal{F}^* = \text{Id on } l^2\,. \end{array}$$

Definition 5.4. *The inner-products on $L^2(\mathbb{T}^n)$ and l^2 are*
$$(u, v)_{L^2(\mathbb{T}^n)} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} u(x)\overline{v(x)}dx$$
and
$$(\hat{u}, \hat{v})_{l^2} = \sum_{k\in\mathbb{Z}^n} \hat{u}_k \overline{\hat{v}_k}\,,$$
respectively.

Parseval's identity shows that $\|u\|_{L^2(\mathbb{T}^n)} = \|\hat{u}\|_{l^2}$.

Definition 5.5. *We set*
$$\mathcal{D}'(\mathbb{T}^n) = [C^\infty(\mathbb{T}^n)]' \quad \textit{and} \quad \mathfrak{s}' = [\mathfrak{s}]'\,.$$
The space $\mathcal{D}'(\mathbb{T}^n)$ is termed the space of periodic distributions.

In the same manner that we extended the Fourier transform from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by duality, we may produce a similar extension to the periodic distributions:
$$\begin{array}{ll} \mathcal{F} : \mathcal{D}'(\mathbb{T}^n) \to \mathfrak{s}' & \mathcal{F}^* : \mathfrak{s}' \to \mathcal{D}'(\mathbb{T}^n) \\ \mathcal{F}^*\mathcal{F} = \text{Id on } \mathcal{D}'(\mathbb{T}^n) & \mathcal{F}\mathcal{F}^* = \text{Id on } \mathfrak{s}'\,. \end{array}$$

Definition 5.6 (Sobolev spaces $H^s(\mathbb{T}^n)$)**.** *For all $s \in \mathbb{R}$, the Hilbert spaces $H^s(\mathbb{T}^n)$ are defined as follows:*
$$H^s(\mathbb{T}^n) = \{u \in \mathcal{D}'(\mathbb{T}^n) \mid \|u\|_{H^s(\mathbb{T}^n)} < \infty\}\,,$$
where the norm on $H^s(\mathbb{T}^n)$ is defined as
$$\|u\|^2_{H^s(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |\hat{u}_k|^2 \langle k\rangle^{2s}\,.$$

The space $(H^s(\mathbb{T}^n), \|\cdot\|_{H^s(\mathbb{T}^n)})$ is a Hilbert space, and we have that
$$H^{-s}(\mathbb{T}^n) = [H^s(\mathbb{T}^n)]'\,.$$

5.2. **The Poisson Integral Formula and the Laplace operator.** For $f : \mathbb{S}^1 \to \mathbb{R}$, denote by $\text{PI}(f)(r,\theta)$ the harmonic function on the unit disk $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ with trace f:
$$\begin{aligned} \Delta\,\text{PI}(f) &= 0 \ \text{ in } \ D \\ \text{PI}(f) &= f \ \text{ on } \ \partial D = \mathbb{S}^1\,. \end{aligned}$$

$\text{PI}(f)$ has an explicit representation via the Fourier series
$$\text{PI}(f)(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta} \quad r < 1, 0 \le \theta < 2\pi\,, \tag{5.1}$$

as well as the integral representation

$$\mathrm{PI}(f)(r,\theta) = \frac{1-r^2}{2\pi}\int_{\mathbb{S}^1} \frac{f(\phi)}{r^2 - 2r\cos(\theta-\phi)+1} d\phi \quad r<1, 0\le\theta<2\pi\,. \tag{5.2}$$

The dominated convergence theorem shows that if $f \in C^0(\mathbb{S}^1)$, then $\mathrm{PI}(f) \in C^\infty(D)\cap C^0(\overline{D})$.

Theorem 5.7. PI *extends to a continuous map from* $H^{k-\frac{1}{2}}(\mathbb{S}^1)$ *to* $H^k(D)$ *for all* $k\in\mathbb{Z}_+$.

Proof. Define $u = \mathrm{PI}(f)$.
Step 1. The case that $k=0$**.** Assume that $f \in H^{-\frac{1}{2}}(\Gamma)$ so that

$$\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \langle k\rangle^{-1} \le M_0 < \infty\,.$$

Since the functions $\{r^{|k|}e^{ik\theta} \ : \ k\in\mathbb{Z}\}$ are orthogonal with respect to the $L^2(D)$ inner-product,

$$\begin{aligned}\|u\|^2_{L^2(D)} &= \int_0^{2\pi}\int_0^1 \left|\sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}\right|^2 r\,dr\,d\theta \\ &\le 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \int_0^1 r^{2|k|+1} dr = \pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 (1+|k|)^{-1} \le \pi \|f\|^2_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,,\end{aligned}$$

where we have used the monotone convergence theorem for the first inequality.
Step 2. The case that $k=1$**.** Next, suppose that $f\in H^{\frac{1}{2}}(\Gamma)$ so that

$$\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \langle k\rangle^{1} \le M_1 < \infty\,.$$

Since we have shown that $u \in L^2(D)$, we must now prove that $u_\theta = \partial_\theta u$ and $u_r = \partial_r u$ are both in $L^2(D)$. Notice that by definition of the Fourier transform and (5.1),

$$\frac{\partial}{\partial\theta}\mathrm{PI}(f) = \mathrm{PI}(f_\theta)\,. \tag{5.3}$$

By definition, $\partial_\theta : H^{\frac{1}{2}}(\mathbb{S}^1) \to H^{-\frac{1}{2}}(\mathbb{S}^1)$ continuously, so that for some constant C,

$$\|f_\theta\|_{H^{-\frac{1}{2}}(\mathbb{S}^1)} \le C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,.$$

It follows from the analysis of **Step 1** and (5.3) that (with $u=\mathrm{PI}(f)$),

$$\|u_\theta\|_{L^2(D)} \le C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,.$$

Next, using the identity (5.1) notice that $|ru_r| = |u_\theta|$. It follows that

$$\|ru_r\|_{L^2(D)} \le C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,. \tag{5.4}$$

By the interior regularity of $-\Delta$ proven in Theorem 6.1, $u_r(r,\theta)$ is smooth on $\{r<1\}$; hence the bound (5.4) implies that, in fact,

$$\|u_r\|_{L^2(D)} \le C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,,$$

and hence

$$\|u\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,,$$

Step 3. The case that $k \geq 2$. Since $f \in H^{k-\frac{1}{2}}(\mathbb{S}^1)$, it follows that

$$\|\partial_\theta^k f\|_{H^{-\frac{1}{2}}(\mathbb{S}^1)} \leq C\|f\|_{H^{k-\frac{1}{2}}(\mathbb{S}^1)}$$

and by repeated application of (5.3), we find that

$$\|u\|_{H^k(D)} \leq C\|f\|_{H^{k-\frac{1}{2}}(\mathbb{S}^1)}\,.$$

□

6. The Laplacian and its regularity

We have studied the regularity properties of the Laplace operator on $D = B(0,1) \subset \mathbb{R}^2$ using the Poisson integral formula. These properties continue to hold on more general open, bounded, C^∞ subsets Ω of $\mathbb{R}^n$.

We revisit the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \tag{6.1a}$$

$$u = f \quad \text{on} \quad \partial\Omega\,. \tag{6.1b}$$

Theorem 6.1. *For $k \in \mathbb{N}$, given $f \in H^{k-\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^k(\Omega)$ to (6.1) satisfying*

$$\|u\|_{H^k(\Omega)} \leq C\|f\|_{H^{k-\frac{1}{2}}(\partial\Omega)}\,, \quad C = C(\Omega)\,.$$

Proof. **Step 1. $k = 1$.** We begin by converting (6.1) to a problem with homogeneous boundary conditions. Using the surjectivity of the trace operator provided by Theorem 4.16, there exists $F \in H^1(\Omega)$ such that $T(F) = f$ on $\partial\Omega$, and $\|F\|_{H^1(\Omega)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}$. Let $U = u - F$; then $U \in H^1(\Omega)$ and by linearity of the trace operator, $T(U) = 0$ on $\partial\Omega$. It follows from Theorem 2.35 that $U \in H_0^1(\Omega)$ and satisfies $-\Delta U = \Delta F$ in $H_0^1(\Omega)$; that is $\langle -\Delta U, v\rangle = \langle \Delta F, v\rangle$ for all $v \in H_0^1(\Omega)$.

According to Remark 2.45, $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, so that $\Delta F \in H^{-1}(\Omega)$; therefore, by Theorem 2.44, there exists a unique weak solution $U \in H_0^1(\Omega)$, satisfying

$$\int_\Omega DU \cdot Dv\, dx = \langle \Delta F, v\rangle \quad \forall v \in H_0^1(\Omega)\,,$$

with

$$\|U\|_{H^1(\Omega)} \leq C\|\Delta F\|_{H^{-1}(\Omega)}\,, \tag{6.2}$$

and hence

$$u = U + F \in H^1(\Omega) \quad \text{and} \quad \|u\|_{H^1(\Omega)} \leq \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}\,.$$

Step 2. $k = 2$. Next, suppose that $f \in H^{1.5}(\partial\Omega)$. Again employing Theorem 4.16, we obtain $F \in H^2(\Omega)$ such that $T(F) = f$ and $\|F\|_{H^2(\Omega)} \leq C\|f\|_{H^{1.5}(\partial\Omega)}$; thus, we see that $\Delta F \in L^2(\Omega)$ and that, in fact,

$$\int_\Omega DU \cdot Dv\, dx = \int_\Omega \Delta F\, v\, dx \quad \forall v \in H_0^1(\Omega)\,. \tag{6.3}$$

We first establish interior regularity. Choose *any* (nonempty) open sets $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and let $\zeta \in C_0^\infty(\Omega_2)$ with $0 \leq \zeta \leq 1$ and $\zeta = 1$ on Ω_1. Let $\epsilon_0 = \min \operatorname{dist}(\operatorname{spt}(\zeta), \partial\Omega_2)/2$. For all $0 < \epsilon < \epsilon_0$, define $U^\epsilon(x) = \eta_\epsilon * U(x)$ for all $x \in \Omega_2$, and set

$$v = -\eta_\epsilon * (\zeta^2 U^\epsilon,_j),_j\,.$$

Then $v \in H_0^1(\Omega)$ and can be used as a test function in (6.3); thus,

$$-\int_\Omega U,_i \ \eta_\epsilon * (\zeta^2 U^\epsilon,_j),_{ji} \, dx = -\int_\Omega U,_i \ \eta_\epsilon * [\zeta^2 U^\epsilon,_{ij} + 2\zeta\zeta,_i U^\epsilon,_j],_j \, dx$$
$$= \int_{\Omega_2} \zeta^2 U^\epsilon,_{ij} U^\epsilon,_{ij} \, dx - 2\int_\Omega \eta_\epsilon * [\zeta\zeta,_i U^\epsilon,_j],_j \ U,_i \, dx \,,$$

and

$$\int_\Omega \Delta F \, v \, dx = -\int_{\Omega_2} \Delta F \, \eta_\epsilon * (\zeta^2 U^\epsilon,_j),_j \, dx = -\int_{\Omega_2} \Delta F \, \eta_\epsilon * [\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j] \, dx \,.$$

By Young's inequality (Theorem 1.53),

$$\|\eta_\epsilon * [\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j]\|_{L^2(\Omega_2)} \le \|\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j\|_{L^2(\Omega_2)};$$

hence, by the Cauchy-Young inequality with δ, Lemma 1.52, for $\delta > 0$,

$$\int_\Omega \Delta F \, v \, dx \le \delta \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} + C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}] \,.$$

Similarly,

$$2\int_\Omega \eta_\epsilon * [\zeta\zeta,_i U^\epsilon,_j],_j \ U,_i \, dx \le \delta \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} + C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}] \,.$$

By choosing $\delta < 1$ and readjusting the constant C_δ, we see that

$$\|D^2 U^\epsilon\|^2_{L^2(\Omega_1)} \le \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} \le C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}]$$
$$\le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \,,$$

the last inequality following from (6.2), and Young's inequality.

Since the right-hand side does not depend on $\epsilon > 0$, there exists a subsequence

$$D^2 U^{\epsilon'} \rightharpoonup \mathcal{W} \ \text{ in } \ L^2(\Omega_1) \,.$$

By Theorem 2.17, $U^\epsilon \to U$ in $H^1(\Omega_1)$, so that $\mathcal{W} = D^2 U$ on Ω_1. As weak convergence is lower semi-continuous, $\|D^2 U\|_{L^2(\Omega_1)} \le C_\epsilon \|\Delta F\|_{L^2(\Omega)}$. As Ω_1 and Ω_2 are arbitrary, we have established that $U \in H^2_{\text{loc}}(\Omega)$ and that

$$\|U\|_{H^2_{\text{loc}}(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} \,.$$

For any $w \in H_0^1(\Omega)$, set $v = \zeta w$ in (6.3). Since $u \in H^2_{\text{loc}}(\Omega)$, we may integrate by parts to find that

$$\int_\Omega (-\Delta U - \Delta F) \, \zeta w \, dx = 0 \ \ \forall w \in H_0^1(\Omega) \,.$$

Since w is arbitrary, and the spt(ζ) can be chosen arbitrarily close to $\partial\Omega$, it follows that for all x in the interior of Ω, we have that

$$-\Delta U(x) = \Delta F(x) \quad \text{for almost every } x \in \Omega \,. \tag{6.4}$$

We proceed to establish the regularity of U all the way to the boundary $\partial\Omega$. Let $\{\mathcal{U}_l\}_{l=1}^K$ denote an open cover of Ω which intersects the boundary $\partial\Omega$, and let $\{\theta_l\}_{l=1}^K$ denote a collection of charts such that

$$\theta_l : B(0, r_l) \to \mathcal{U}_l \text{ is a } C^\infty \text{ diffeomorphism} \,,$$
$$\det D\theta_l = 1 \,,$$
$$\theta_l(B(0, r_l) \cap \{x_n = 0\}) \to \mathcal{U}_l \cap \partial\Omega \,,$$
$$\theta_l(B(0, r_l) \cap \{x_n > 0\}) \to \mathcal{U}_l \cap \Omega \,.$$

Let $0 \leq \zeta_l \leq 1$ in $C_0^\infty(\mathcal{U}_l)$ denote a partition of unity subordinate to the open covering $\mathcal{U}_l$, and define the horizontal convolution operator, smoothing functions defined on $\mathbb{R}^n$ in the first $1, ..., n-1$ directions, as follows:

$$\rho_\epsilon *_h F(x_h, x_n) = \int_{\mathbb{R}^{n-1}} \rho_\epsilon(x_h - y_h) F(y_h, x_n) dy_h \,,$$

where $\rho_\epsilon(x_h) = \epsilon^{-(n-1)}\rho(x_h/\epsilon)$, ρ the standard mollifier on $\mathbb{R}^{n-1}$, and $x_h = (x_1, ..., x_{n-1})$. Let α range from 1 to $n-1$, and substitute the test function

$$v = -\left(\rho_\epsilon *_h [(\zeta_l \circ \theta_l)^2 \rho_\epsilon *_h (U \circ \theta_l),_\alpha],_\alpha\right) \circ \theta_l^{-1} \in H_0^1(\Omega)$$

into (6.3), and use the change of variables formula to obtain the identity

$$\int_{B_+(0,r_l)} A_i^k (U \circ \theta_l),_k \, A_i^j (v \circ \theta_l),_j \; dx = \int_{B_+(0,r_l)} (\Delta F) \circ \theta_l \, v \circ \theta_l \, dx \,, \tag{6.5}$$

where the C^∞ matrix $A(x) = [D\theta_l(x)]^{-1}$ and $B_+(0, r_l) = B(0, r_l) \cap \{x_n > 0\}$. We define

$$U^l = U \circ \theta_l \,, \text{ and denote the horizontal convolution operator by } H_\epsilon = \rho_\epsilon *_h \,.$$

Then, with $\xi_l = \zeta_l \circ \theta_l$, we can rewrite the test function as

$$v \circ \theta_l = -H_\epsilon[\xi_l^2 H_\epsilon U^l,_\alpha],_\alpha \,.$$

Since differentiation commutes with convolution, we have that

$$(v \circ \theta_l),_j = -H_\epsilon(\xi_l^2 H_\epsilon U^l,_{j\alpha}),_\alpha - 2H_\epsilon(\xi_l \xi_l,_j \, H_\epsilon U^l,_\alpha),_\alpha \,,$$

and we can express the left-hand side of (6.5) as

$$\int_{B_+(0,r_l)} A_i^k (U \circ \theta_l),_k \, A_i^j (v \circ \theta_l),_j \; dx = \mathcal{I}_1 + \mathcal{I}_2 \,,$$

where

$$\mathcal{I}_1 = -\int_{B_+(0,r_l)} A_i^j A_i^k U^l,_k \; H_\epsilon(\xi_l^2 H_\epsilon U^l,_{j\alpha}),_\alpha \; dx \,,$$
$$\mathcal{I}_2 = -2\int_{B_+(0,r_l)} A_i^j A_i^k U^l,_k \; H_\epsilon(\xi_l \xi_l,_j \, H_\epsilon U^l,_\alpha),_\alpha \; dx \,.$$

Next, we see that

$$\mathcal{I}_1 = \int_{B_+(0,r_l)} [H_\epsilon(A_i^j A_i^k U^l,_k)],_\alpha \; (\xi_l^2 H_\epsilon U^l,_{j\alpha}) \, dx = \mathcal{I}_{1a} + \mathcal{I}_{1b} \,,$$

where

$$\mathcal{I}_{1a} = \int_{B_+(0,r_l)} (A_i^j A_i^k H_\epsilon U^l,_k),_\alpha \; \xi_l^2 H_\epsilon U^l,_{j\alpha} \; dx \,,$$
$$\mathcal{I}_{1b} = \int_{B_+(0,r_l)} ([H_\epsilon, A_i^j A_i^k] U^l,_k),_\alpha \; \xi_l^2 H_\epsilon U^l,_{j\alpha} \; dx \,,$$

and where

$$[H_\epsilon, A_i^j A_i^k] U^l,_k = H_\epsilon(A_i^j A_i^k U^l,_k) - A_i^j A_i^k \, H_\epsilon U^l,_k \tag{6.6}$$

denotes the commutator of the horizontal convolution operator and multiplication. The integral $\mathcal{I}_{1a}$ produces the positive sign-definite term which will allow us to build the global regularity of U, as well as an error term:

$$\mathcal{I}_{1a} = \int_{B_+(0,r_l)} [\xi_l^2 A_i^j A_i^k H_\epsilon U^l,_{k\alpha} \ H_\epsilon U^l,_{j\alpha} + (A_i^j A_i^k),_\alpha \ H_\epsilon U^l,_k \ \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \, ;$$

thus, together with the right hand-side of (6.5), we see that

$$\int_{B_+(0,r_l)} \xi_l^2 A_i^j A_i^k H_\epsilon U^l,_{k\alpha} \ H_\epsilon U^l,_{j\alpha} \ dx \le \left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha \ H_\epsilon U^l,_k \ \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \right| + |\mathcal{I}_{1b}| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F) \circ \theta_l \, v \circ \theta_l \, dx \right| .$$

Since each θ_l is a C^∞ diffeomorphism, it follows that the matrix $A\,A^T$ is positive definite: there exists $\lambda > 0$ such that

$$\lambda |Y|^2 \le A_i^j A_i^k Y_j Y_k \quad \forall Y \in \mathbb{R}^n \, .$$

It follows that

$$\lambda \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx \le \left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha \ H_\epsilon U^l,_k \ \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \right| + |\mathcal{I}_{1b}| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F) \circ \theta_l \, v \circ \theta_l \, dx \right| ,$$

where $D = (\partial_{x_1}, ..., \partial_{x_n})$ and $\bar p = (\partial_{x_1}, ..., \partial_{x_{n-1}})$. Application of the Cauchy-Young inequality with $\delta > 0$ shows that

$$\left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha \ H_\epsilon U^l,_k \ \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \right| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F) \circ \theta_l \, v \circ \theta_l \, dx \right| \le \delta \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx + C_\delta \|\Delta F\|^2_{L^2(\Omega)} \, .$$

It remains to establish such an upper bound for $|\mathcal{I}_{1b}|$.

To do so, we first establish a pointwise bound for (6.6): for $\mathcal{A}^{jk} = A_i^j A_i^k$,

$$[H_\epsilon, A_i^j A_i^k] U^l,_k (x) = \int_{B(x_h,\epsilon)} \rho_\epsilon(x_h - y_h) [\mathcal{A}^{jk}(y_h, x_n) - \mathcal{A}^{jk}(x_h, x_n)] U^l,_k (y_h, x_n) \, dy_h$$

By Morrey's inequality, $|[\mathcal{A}^{jk}(y_h, x_n) - \mathcal{A}^{jk}(x_h, x_n)]| \le C\epsilon \|\mathcal{A}\|_{W^{1,\infty}(B_+(0,r_l))}$. Since

$$\partial_{x_\alpha} \rho_\epsilon(x_h - y_h) = \frac{1}{\epsilon^2} \rho' \left(\frac{x - h - y_h}{\epsilon} \right) ,$$

we see that

$$\left| \partial_{x_\alpha} \left([H_\epsilon, A_i^j A_i^k] U^l,_k \right)(x) \right| \le C \int_{B(x_h,\epsilon)} \frac{1}{\epsilon} \rho' \left(\frac{x - h - y_h}{\epsilon} \right) |U^l,_k (y_h, x_n)| \, dy_h$$

and hence by Young's inquality,

$$\left\| \partial_{x_\alpha} \left([H_\epsilon, A_i^j A_i^k] U^l,_k \right) \right\|_{L^2(B_+(0,r_l))} \le C \|U\|_{H^1(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} \, .$$

It follows from the Cauchy-Young inequality with $\delta > 0$ that

$$|\mathcal{I}_{1b}| \le \delta \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx + C_\delta \|\Delta F\|^2_{L^2(\Omega)} \, .$$

By choosing $2\delta < \lambda$, we obtain the estimate

$$\int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \, .$$

Since the right hand-side is independent of ϵ, we find that

$$\int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \, . \tag{6.7}$$

From (6.4), we know that $\Delta U(x) = \Delta F(x)$ for a.e. $x \in \mathcal{U}_l$. By the chain-rule this means that almost everywhere in $B_+(0, r_l)$,

$$-\mathcal{A}^{jk} U^l,_{kj} = \mathcal{A}^{jk},_j U^l,_k + \Delta F \circ \theta_l \, ,$$

or equivalently,

$$-\mathcal{A}^{nn} U^l_{nn} = \mathcal{A}^{j\alpha} U^l,_{\alpha j} + \mathcal{A}^{\beta k} U^l,_{k\beta} + \mathcal{A}^{jk},_j U^l,_k + \Delta F \circ \theta_l \, .$$

Since $\mathcal{A}^{nn} > 0$, it follows from (6.7) that

$$\int_{B_+(0,r_l)} \xi_l^2 |D^2 U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \, . \tag{6.8}$$

Summing over l from 1 to K and combining with our interior estimates, we have that

$$\|u\|_{H^2(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} \, .$$

Step 3. $k \ge 3$. At this stage, we have obtained a pointwise solution $U \in H^2(\Omega) \cap H^1_0(\Omega)$ to $\Delta U = \Delta F$ in Ω, and $\Delta F \in H^{k-1}$. We differentiate this equation r times until $D^r \Delta F \in L^2(\Omega)$, and then repeat Step 2. □

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