Problem Sets for Linear Algebra in Twenty Five Lectures

February 7, 2012

Selected problems for students to hand in.

Contents

1 Problems: What is Linear Algebra

1. Let M be a matrix and u and v vectors:
   $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v = \begin{pmatrix} x \\ y \end{pmatrix}, u = \begin{pmatrix} w \\ z \end{pmatrix}.$

   (a) *Propose* a definition for $u + v$.

   (b) *Check* that your definition obeys $Mv + Mu = M(u + v)$.

2. *Matrix Multiplication:* Let M and N be matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } N = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

and v a vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix}.$$

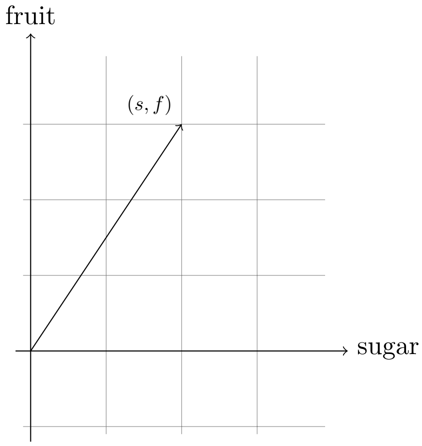
Compute the vector Nv using the rule given above. Now multiply this vector by the matrix M, *i.e.*, compute the vector $M(Nv)$.

Next recall that multiplication of ordinary numbers is associative, namely the order of brackets does not matter: $(xy)z = x(yz)$. Let us try to demand the same property for matrices and vectors, that is

$$M(Nv) = (MN)v.$$

We need to be careful reading this equation because Nv is a vector and so is $M(Nv)$. Therefore the right hand side, $(MN)v$ should also be a vector. This means that MN must be a matrix; in fact it is the matrix obtained by multiplying the matrices M and N. Use your result for $M(Nv)$ to find the matrix MN.

3. Pablo is a nutritionist who knows that oranges always have twice as much sugar as apples. When considering the sugar intake of schoolchildren eating a barrel of fruit, he represents the barrel like so:



Find a linear transformation relating Pablo's representation to the one in the lecture. Write your answer as a matrix.

Hint: Let λ represent the amount of sugar in each apple.

Hint

4. There are methods for solving linear systems other than Gauss' method. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. That step is repeated until there is an equation with only one variable. From that, the first number in the solution is derived, and then back-substitution can be done. This method takes longer than Gauss' method, since it involves more arithmetic operations, and is also more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

   (a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

   (b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

   What extra step must a user of this method take to avoid erroneously concluding a system has a solution?

2 Problems: Gaussian Elimination

1. State whether the following augmented matrices are in RREF and compute their solution sets.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

2. Show that this pair of augmented matrices are row equivalent, assuming $ad - bc \neq 0$:

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-bc} \\ 0 & 1 & \frac{af-ce}{ad-bc} \end{array}\right)$$

3. Consider the augmented matrix: $\left(\begin{array}{cc|c} 2 & -1 & 3 \\ -6 & 3 & 1 \end{array}\right)$

Give a *geometric* reason why the associated system of equations has no solution. (Hint, plot the three vectors given by the columns of this augmented matrix in the plane.) Given a general augmented matrix

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right),$$

can you find a condition on the numbers a, b, c and d that create the geometric condition you found?

4. List as many operations on augmented matrices that *preserve* row equivalence as you can. Explain your answers. Give examples of operations that break row equivalence.

5. Row equivalence of matrices is an example of an *equivalence relation*. Recall that a relation $\sim$ on a set of objects U is an equivalence relation if the following three properties are satisfied:

- Reflexive: For any $x \in U$, we have $x \sim x$.
- Symmetric: For any $x, y \in U$, if $x \sim y$ then $y \sim x$.
- Transitive: For any x, y and $z \in U$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

(For a fuller discussion of equivalence relations, see Homework 0, Problem 4)

Show that row equivalence of augmented matrices is an equivalence relation.

`Hints for Questions 4 and 5`

3 Problems: Elementary Row Operations

1. (Row Equivalence)

   (a) Solve the following linear system using Gauss-Jordan elimination:

$$\begin{aligned}
2x_1 + 5x_2 - 8x_3 + 2x_4 + 2x_5 &= 0 \\
6x_1 + 2x_2 - 10x_3 + 6x_4 + 8x_5 &= 6 \\
3x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 &= 6 \\
3x_1 + 1x_2 - 5x_3 + 3x_4 + 4x_5 &= 3 \\
6x_1 + 7x_2 - 3x_3 + 6x_4 + 9x_5 &= 9
\end{aligned}$$

   Be sure to set your work out carefully with equivalence signs $\sim$ *between each step, labeled by the row operations you performed.*

   (b) Check that the following two matrices are row-equivalent:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 2 & 9 & 6 & 0 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 0 & -1 & 8 & 20 \\ 4 & 18 & 12 & 0 \end{array}\right)$$

   Now remove the third column from each matrix, and show that the resulting two matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 2 & 9 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 20 \\ 4 & 18 & 0 \end{array}\right)$$

   Now remove the fourth column from each of the original two matrices, and show that the resulting two matrices, viewed as augmented matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 9 & 6 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 8 \\ 4 & 18 & 12 \end{array}\right)$$

   Explain why row-equivalence is never affected by removing columns.

   (c) Check that the matrix $\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 3 & 13 & 9 \\ 4 & 17 & 20 \end{array}\right)$ has no solutions. If you remove one of the rows of this matrix, does the new matrix have any solutions? In general, can row equivalence be affected by removing rows? Explain why or why not.

2. (Gaussian Elimination) Another method for solving linear systems is to use row operations to bring the augmented matrix to row echelon form. In row echelon form, the pivots are not necessarily set to one, and we only require that all entries left of the pivots are zero, not necessarily entries above a pivot. Provide a counterexample to show that row echelon form is not unique.

   Once a system is in row echelon form, it can be solved by "back substitution." Write the following row echelon matrix as a system of equations, then solve the system using back-substitution.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

3. Explain why the linear system has no solutions:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

For which values of k does the system below have a solution?

$$\begin{array}{rcrcrcr} x & - & 3y & & & = & 6 \\ x & & & + & 3\,z & = & -\,3 \\ 2x & + & ky & + & (3-k)z & = & 1 \end{array}$$

`Hint for question 3`

4 Problems: Solution Sets for Systems of Linear Equations

1. Let $f(X) = MX$ where

$$M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Suppose that α is any number. Compute the following four quantities:

$$\alpha X\,,\ f(X)\,,\ \alpha f(X) \text{ and } f(\alpha X)\,.$$

Check your work by verifying that

$$\alpha f(X) = f(\alpha X)\,.$$

Now explain why the result checked in the Lecture, namely

$$f(X + Y) = f(X) + f(Y)\,,$$

and your result $f(\alpha X) = \alpha f(X)$ together imply

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)\,.$$

2. Write down examples of augmented matrices corresponding to each of the five types of solution sets for systems of equations with three unknowns.

3. Let

$$M = \begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_k \\ a^2_1 & a^2_2 & \cdots & a^2_k \\ \vdots & \vdots & & \vdots \\ a^r_1 & a^r_2 & \cdots & a^r_k \end{pmatrix}, \qquad X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}$$

Propose a rule for MX so that $MX = 0$ is equivalent to the linear system:

$$\begin{array}{cccc} a^1_1x^1 & +a^1_2x^2 & \cdots +a^1_kx^k & = 0 \\ a^2_1x^1 & +a^2_2x^2 & \cdots +a^2_kx^k & = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a^r_1x^1 & +a^r_2x^2 & \cdots +a^r_kx^k & = 0 \end{array}$$

Show that your rule for multiplying a matrix by a vector obeys the linearity property.

Note that in this problem, x^2 does not denote the square of x. Instead x^1, x^2, x^3, etc... denote different variables. Although confusing at first, this notation was invented by Albert Einstein who noticed that quantities like $a^2_1x^1 + a^2_2x^2 \cdots + a^2_kx^k$ could be written in summation notation as $\sum_{j=1}^k a^2_jx^j$. Here j is called a summation index. Einstein observed that you could even drop the summation sign $\sum$ and simply write $a^2_jx^j$.

Problem 3 hint

4. Use the rule you developed in the problem 3 to compute the following products

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 14 \\ 21 \\ 35 \\ 62 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 42 & 97 & 2 & -23 & 46 \\ 0 & 1 & 3 & 1 & 0 & 33 \\ 11 & \pi & 1 & 0 & 46 & 29 \\ -98 & 12 & 0 & 33 & 99 & 98 \\ \log 2 & 0 & \sqrt{2} & 0 & e & 23 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now that you are good at multiplying a matrix with a column vector, try your hand at a product of two matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hint, to do this problem view the matrix on the right as three column vectors next to one another.

5. The *standard basis vector* e_i is a column vector with a one in the ith row, and zeroes everywhere else. Using the rule for multiplying a matrix times a vector in problem 3, find a simple rule for multiplying Me_i, where M is the general matrix defined there.

5 Problems: Vectors in Space, n-Vectors

1. When he was young, Captain Conundrum mowed lawns on weekends to help pay his college tuition bills. He charged his customers according to the size of their lawns at a rate of 5¢ per square foot and meticulously kept a record of the areas of their lawns in an ordered list:

$$A = (200, 300, 50, 50, 100, 100, 200, 500, 1000, 100)\,.$$

He also listed the number of times he mowed each lawn in a given year, for the year 1988 that ordered list was

$$f = (20, 1, 2, 4, 1, 5, 2, 1, 10, 6)\,.$$

(a) Pretend that A and f are vectors and compute $A \cdot f$.

(b) What quantity does the dot product $A \cdot f$ measure?

(c) How much did Captain Conundrum earn from mowing lawns in 1988? Write an expression for this amount in terms of the vectors A and f.

(d) Suppose Captain Conundrum charged different customers different rates. How could you modify the expression in part 1c to compute the Captain's earnings?

2. (2) Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and one of the coordinate axes.

(3) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the coordinate axes.

(n) Find the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes.

(∞) What is the limit as $n \to \infty$ of the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes?

3. Consider the matrix $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and the vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

   (a) Sketch X and MX in $\mathbb{R}^2$ for several values of X and θ.

   (b) Compute $\frac{||MX||}{||X||}$ for arbitrary values of X and θ.

   (c) Explain your result for (b) and describe the action of M geometrically.

4. Suppose in $\mathbb{R}^2$ I measure the x direction in inches and the y direction in miles. Approximately what is the real-world angle between the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$? What is the angle between these two vectors according to the dot-product? Give a definition for an inner product so that the angles produced by the inner product are the actual angles between vectors.

5. (Lorentzian Strangeness). For this problem, consider $\mathbb{R}^n$ with the Lorentzian inner product and metric defined above.

   (a) Find a non-zero vector in two-dimensional Lorentzian space-time with zero length.

   (b) Find and sketch the collection of all vectors in two-dimensional Lorentzian space-time with zero length.

   (c) Find and sketch the collection of all vectors in three-dimensional Lorentzian space-time with zero length.

The Story of Your Life

6 Problems: Vector Spaces

1. Check that $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\} = \mathbb{R}^2$ with the usual addition and scalar multiplication is a vector space.

2. Check that the complex numbers $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$ form a vector space over $\mathbb{C}$. Make sure you state carefully what your rules for vector addition and scalar multiplication are. Also, explain what would happen if you used $\mathbb{R}$ as the base field (try comparing to problem 1).

3. (a) Consider the set of convergent sequences, with the same addition and scalar multiplication that we defined for the space of sequences:

$$V = \left\{ f | f : \mathbb{N} \to \mathbb{R}, \lim_{n\to\infty} f \in \mathbb{R} \right\}$$

Is this still a vector space? Explain why or why not.

(b) Now consider the set of divergent sequences, with the same addition and scalar multiplication as before:

$$V = \left\{ f | f : \mathbb{N} \to \mathbb{R}, \lim_{n\to\infty} f \text{ does not exist or is } \pm\infty \right\}$$

Is this a vector space? Explain why or why not.

4. Consider the set of 2×4 matrices:

$$V = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} | a, b, c, d, e, f, g, h \in \mathbb{C} \right\}$$

Propose definitions for addition and scalar multiplication in V. Identify the zero vector in V, and check that every matrix has an additive inverse.

5. Let $P_3^{\mathbb{R}}$ be the set of polynomials with real coefficients of degree three or less.

- Propose a definition of addition and scalar multiplication to make $P_3^{\mathbb{R}}$ a vector space.
- Identify the zero vector, and find the additive inverse for the vector $-3 - 2x + x^2$.
- Show that $P_3^{\mathbb{R}}$ is not a vector space over $\mathbb{C}$. Propose a small change to the definition of $P_3^{\mathbb{R}}$ to make it a vector space over $\mathbb{C}$.

Problem 5 hint

7 Problems: Linear Transformations

1. Show that the pair of conditions:

   (i) $L(u+v) = L(u) + L(v)$

   (ii) $L(cv) = cL(v)$

   is equivalent to the single condition:

   (iii) $L(ru+sv) = rL(u) + sL(v)$.

   Your answer should have two parts. Show that (i,ii)⇒(iii), and then show that (iii)⇒(i,ii).

2. Let P_n be the space of polynomials of degree n or less in the variable t. Suppose L is a linear transformation from $P_2 \to P_3$ such that $L(1) = 4, L(t) = t^3$, and $L(t^2) = t - 1$.

- Find $L(1 + t + 2t^2)$.
- Find $L(a + bt + ct^2)$.
- Find all values a, b, c such that $L(a + bt + ct^2) = 1 + 3t + 2t^3$.

Hint

3. Show that integration is a linear transformation on the vector space of polynomials. What would a matrix for integration look like? Be sure to think about what to do with the constant of integration.

`Finite degree example`

8 Problems: Matrices

1. Compute the following matrix products

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix} \begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix} \begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix},$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & \frac{2}{3} & -\frac{2}{3} \\ 6 & \frac{5}{3} & -\frac{2}{3} \\ 12 & -\frac{16}{3} & \frac{10}{3} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix}.$$

2. Let's prove the theorem $(MN)^T = N^T M^T$.

   Note: the following is a common technique for proving matrix identities.

   (a) Let $M = (m^i_j)$ and let $N = (n^i_j)$. Write out a few of the entries of each matrix in the form given at the beginning of this chapter.

   (b) Multiply out MN and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of MN?

   (c) Take the transpose $(MN)^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $(MN)^T$?

   (d) Take the transposes N^T and M^T and write out a few of their entries in the same form as in part a.

   (e) Multiply out $N^T M^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $N^T M^T$?

   (f) Show that the answers you got in parts c and e are the same.

3. Let M be any $m \times n$ matrix. Show that $M^T M$ and $M M^T$ are symmetric. (Hint: use the result of the previous problem.) What are their sizes?

4. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ be column vectors. Show that the dot product $x \bullet y = x^T \ \mathbb{1} \ y$.

5. Above, we showed that *left* multiplication by an $r \times s$ matrix N was a linear transformation $M_k^s \xrightarrow{N} M_k^r$. Show that *right* multiplication by a $k \times m$ matrix R is a linear transformation $M_k^s \xrightarrow{R} M_m^s$. In other words, show that right matrix multiplication obeys linearity.

`Problem hint`

6. Explain what happens to a matrix when:

   (a) You multiply it on the left by a diagonal matrix.

   (b) You multiply it on the right by a diagonal matrix.

   Give a few simple examples before you start explaining.

9 Problems: Properties of Matrices

1. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find AA^T and A^TA. What can you say about matrices MM^T and M^TM in general? Explain.

2. Compute $\exp(A)$ for the following matrices:

- $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
- $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$
- $A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

Hint

3. Suppose $ad - bc \neq 0$, and let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) Find a matrix M^{-1} such that $MM^{-1} = I$.

   (b) Explain why your result explains what you found in a previous homework exercise.

   (c) Compute $M^{-1}M$.

4. Let $M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$. Divide M into named blocks, and then multiply blocks to compute M^2.

10 Problems: Inverse Matrix

1. Find formulas for the inverses of the following matrices, when they are not singular:

   (a) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

   When are these matrices singular?

2. Write down all 2×2 bit matrices and decide which of them are singular. For those which are not singular, pair them with their inverse.

3. Let M be a square matrix. Explain why the following statements are equivalent:

   (a) $MX = V$ has a *unique* solution for every column vector V.

   (b) M is non-singular.

   (In general for problems like this, think about the key words:

   First, suppose that there is some column vector V such that the equation $MX = V$ has two distinct solutions. Show that M must be singular; that is, show that M can have no inverse.

   Next, suppose that there is some column vector V such that the equation $MX = V$ has no solutions. Show that M must be singular.

   Finally, suppose that M is non-singular. Show that no matter what the column vector V is, there is a unique solution to $MX = V$.)

`Hints for Problem 3`

4. *Left and Right Inverses:* So far we have only talked about inverses of square matrices. This problem will explore the notion of a left and right inverse for a matrix that is not square. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) Compute:

i. AA^T,

ii. $\left(AA^T\right)^{-1}$,

iii. $B := A^T\left(AA^T\right)^{-1}$

(b) Show that the matrix B above is a *right inverse* for A, *i.e.*, verify that

$$AB = I\,.$$

(c) Does BA make sense? (Why not?)

(d) Let A be an $n \times m$ matrix with $n > m$. Suggest a formula for a left inverse C such that

$$CA = I$$

Hint: you may assume that A^TA has an inverse.

(e) Test your proposal for a left inverse for the simple example

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,,$$

(f) True or false: Left and right inverses are unique. If false give a counterexample.

`Left and Right Inverses`

11 Problems: LU Decomposition

1. Consider the linear system:

$$\begin{array}{lllll} x^1 & & & & = v^1 \\ l^2_1 x^1 & +x^2 & & & = v^2 \\ \vdots & & & & \vdots \\ l^n_1 x^1 & +l^n_2 x^2 & + \cdots + & x^n & = v^n \end{array}$$

   i. Find x^1.

   ii. Find x^2.

   iii. Find x^3.

   k. Try to find a formula for x^k. Don't worry about simplifying your answer.

2. Let $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be a square $n \times n$ block matrix with W invertible.

i. If W has r rows, what size are X, Y, and Z?

ii. Find a UDL decomposition for M. In other words, fill in the stars in the following equation:

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}$$

12 Problems: Elementary Matrices and Determinants

1. Let $M = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{pmatrix}$. Use row operations to put M into *row echelon form*. For simplicity, assume that $m_1^1 \neq 0 \neq m_1^1 m_2^2 - m_1^2 m_2^1$.

   Prove that M is non-singular if and only if:

$$m_1^1 m_2^2 m_3^3 - m_1^1 m_3^2 m_2^3 + m_2^1 m_3^2 m_1^3 - m_2^1 m_1^2 m_3^3 + m_3^1 m_1^2 m_2^3 - m_3^1 m_2^2 m_1^3 \neq 0$$

2. (a) What does the matrix $E_2^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ do to $M = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ under left multiplication? What about right multiplication?

   (b) Find elementary matrices $R^1(\lambda)$ and $R^2(\lambda)$ that respectively multiply rows 1 and 2 of M by λ but otherwise leave M the same under left multiplication.

   (c) Find a matrix $S_2^1(\lambda)$ that adds a multiple λ of row 2 to row 1 under left multiplication.

3. Let M be a matrix and $S^i_j M$ the same matrix with rows i and j switched. Explain every line of the series of equations proving that $\det M = -\det(S^i_j M)$.

4. This problem is a "hands-on" look at why the property describing the parity of permutations is true.

The *inversion number* of a permutation σ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$; it's the number of "numbers that appear left of smaller numbers" in the permutation. For example, for the permutation $\rho = [4, 2, 3, 1]$, the inversion number is 5. The number 4 comes before $2, 3$, and 1, and 2 and 3 both come before 1.

Given a permutation σ, we can make a new permutation $\tau_{i,j}\sigma$ by exchanging the ith and jth entries of σ.

(a) What is the inversion number of the permutation $\mu = [1, 2, 4, 3]$ that exchanges 4 and 3 and leaves everything else alone? Is it an even or an odd permutation?

(b) What is the inversion number of the permutation $\rho = [4, 2, 3, 1]$ that exchanges 1 and 4 and leaves everything else alone? Is it an even or an odd permutation?

(c) What is the inversion number of the permutation $\tau_{1,3}\mu$? Compare the parity[1] of μ to the parity of $\tau_{1,3}\mu$.

(d) What is the inversion number of the permutation $\tau_{2,4}\rho$? Compare the parity of ρ to the parity of $\tau_{2,4}\rho$.

(e) What is the inversion number of the permutation $\tau_{3,4}\rho$? Compare the parity of ρ to the parity of $\tau_{3,4}\rho$.

`Problem 4 hints`

5. (Extra credit) Here we will examine a (very) small set of the general properties about permutations and their applications. In particular, we will show that one way to compute the sign of a permutation is by finding the inversion number N of σ and we have

$$\operatorname{sgn}(\sigma) = (-1)^N.$$

For this problem, let $\mu = [1, 2, 4, 3]$.

(a) Show that every permutation σ can be sorted by only taking simple (adjacent) transpositions s_i where s_i interchanges the numbers in position i and $i + 1$ of a permutation σ (in our other notation $s_i = \tau_{i,i+1}$). For example $s_2\mu = [1, 4, 2, 3]$, and to sort μ we have $s_3\mu = [1, 2, 3, 4]$.

(b) We can compose simple transpositions together to represent a permutation (note that the sequence of compositions is not unique), and these are associative, we have an identity (the trivial permutation where the list is in order or we do nothing on our list), and we have an inverse since it is clear that $s_i s_i \sigma = \sigma$. Thus permutations of $[n]$ under composition are an example of a group. However note that not all simple transpositions commute with each other since

$$s_1 s_2[1, 2, 3] = s_1[1, 3, 2] = [3, 1, 2]$$
$$s_2 s_1[1, 2, 3] = s_2[2, 1, 3] = [2, 3, 1]$$

(you will prove here when simple transpositions commute). When we consider our initial permutation to be the trivial permutation $e = [1, 2, \ldots, n]$, we do not write it; for example $s_i \equiv s_i e$ and $\mu = s_3 \equiv s_3 e$. This is analogous to not writing 1 when multiplying. Show that $s_i s_i = e$ (in shorthand $s_i^2 = e$), $s_{i+1}s_i s_{i+1} = s_i s_{i+1} s_i$ for all i, and s_i and s_j commute for all $|i - j| \geq 2$.

(c) Show that every way of expressing σ can be obtained from using the relations proved in part 5b. In other words, show that for any expression w of simple transpositions representing the trivial permutation e, using the proved relations.

Hint: Use induction on n. For the induction step, follow the path of the $(n + 1)$-th strand by looking at $s_n s_{n-1} \cdots s_k s_{k\pm1} \cdots s_n$ and argue why you can write this as a subexpression for any expression of e. Consider using diagrams of these paths to help.

[1] The *parity* of an integer refers to whether the integer is even or odd. Here the parity of a permutation μ refers to the parity of its inversion number.

(d) The simple transpositions acts on an n-dimensional vector space V by $s_i v = E^i_{i+1} v$ (where E^i_j is an elementary matrix) for all vectors $v \in V$. Therefore we can just represent a permutation σ as the matrix M_σ[2], and we have $\det(M_{s_i}) = \det(E^i_{i+1}) = -1$. Thus prove that $\det(M_\sigma) = (-1)^N$ where N is a number of simple transpositions needed to represent σ as a permutation. You can assume that $M_{s_i s_j} = M_{s_i} M_{s_j}$ (it is not hard to prove) and that $\det(AB) = \det(A)\det(B)$ from Chapter **??**.

Hint: You to make sure $\det(M_\sigma)$ *is well-defined since there are infinite ways to represent* σ *as simple transpositions.*

(e) Show that $s_{i+1}s_i s_{i+1} = \tau_{i,i+2}$, and so give one way of writing $\tau_{i,j}$ in terms of simple transpositions? Is $\tau_{i,j}$ an even or an odd permutation? What is $\det(M_{\tau_{i,j}})$? What is the inversion number of $\tau_{i,j}$?

(f) The minimal number of simple transpositions needed to express σ is called the *length* of σ; for example the length of μ is 1 since $\mu = s_3$. Show that the length of σ is equal to the inversion number of σ.

Hint: Find an procedure which gives you a new permutation σ' *where* $\sigma = s_i\sigma'$ *for some* i *and the inversion number for* σ' *is 1 less than the inversion number for* σ.

(g) Show that $(-1)^N = \operatorname{sgn}(\sigma) = \det(M_\sigma)$, where σ is a permutation with N inversions. Note that this immediately implies that $\operatorname{sgn}(\sigma\rho) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)$ for any permutations σ and ρ.

[2] Often people will just use σ for the matrix when the context is clear.

13 Problems: Elementary Matrices and Determinants II

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Compute the following:

   (a) $\det M$.

   (b) $\det N$.

   (c) $\det(MN)$.

   (d) $\det M \det N$.

   (e) $\det(M^{-1})$ assuming $ad - bc \neq 0$.

   (f) $\det(M^T)$

   (g) $\det(M + N) - (\det M + \det N)$. Is the determinant a linear transformation from square matrices to real numbers? Explain.

2. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Write M as a product of elementary row matrices times $\mathrm{RREF}(M)$.

3. Find the inverses of each of the elementary matrices, $E^i_j, R^i(\lambda), S^i_j(\lambda)$. Make sure to show that the elementary matrix times its inverse is actually the identity.

4. (Extra Credit) Let e^i_j denote the matrix with a 1 in the i-th row and j-th column and 0's everywhere else, and let A be an arbitrary 2×2 matrix. Compute $\det(A + tI_2)$, and what is first order term (the coefficient of t)? Can you express your results in terms of $\operatorname{tr}(A)$? What about the first order term in $\det(A + tI_n)$ for any arbitrary $n \times n$ matrix A in terms of $\operatorname{tr}(A)$?

   We note that the result of $\det(A + tI_2)$ is what is known as the *characteristic polynomial* from Chapter **??** and is a polynomial in the variable t.

5. (Extra Credit: (Directional) Derivative of the Determinant) Notice that $\det\colon \mathbb{M}_n \to \mathbb{R}$ where $\mathbb{M}_n$ is the vector space of all $n \times n$ matrices, and so we can take directional derivatives of det. Let A be an arbitrary $n \times n$ matrix, and for all i and j compute the following:

   (a) $\lim_{t\to 0}(\det(I_2 + te^i_j) - \det(I_2))/t$

   (b) $\lim_{t\to 0}(\det(I_3 + te^i_j) - \det(I_3))/t$

   (c) $\lim_{t\to 0}(\det(I_n + te^i_j) - \det(I_n))/t$

   (d) $\lim_{t\to 0}(\det(I_n + At) - \det(I_n))/t$

   (Recall that what you are calculating is the directional derivative in the e^i_j and A directions.) Can you express your results in terms of the trace function?

   Hint: Use the results from Problem 4 and what you know about the derivatives of polynomials evaluated at 0.

14 Problems: Properties of the Determinant

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show:

$$\det M = \frac{1}{2}(\operatorname{tr} M)^2 - \frac{1}{2}\operatorname{tr}(M^2)$$

   Suppose M is a 3×3 matrix. Find and verify a similar formula for $\det M$ in terms of $\operatorname{tr}(M^3)$, $(\operatorname{tr} M)(\operatorname{tr}(M^2))$, and $(\operatorname{tr} M)^3$.

2. Suppose $M = LU$ is an LU decomposition. Explain how you would efficiently compute $\det M$ in this case.

3. In computer science, the *complexity* of an algorithm is computed (roughly) by counting the number of times a given operation is performed. Suppose adding or subtracting any two numbers takes a seconds, and multiplying two numbers takes m seconds. Then, for example, computing $2 \cdot 6 - 5$ would take $a + m$ seconds.

(a) How many additions and multiplications does it take to compute the determinant of a general 2×2 matrix?

(b) Write a formula for the number of additions and multiplications it takes to compute the determinant of a general $n \times n$ matrix using the definition of the determinant. Assume that finding and multiplying by the sign of a permutation is free.

(c) How many additions and multiplications does it take to compute the determinant of a general 3×3 matrix using expansion by minors? Assuming $m = 2a$, is this faster than computing the determinant from the definition?

15 Problems: Subspaces and Spanning Sets

1. (Subspace Theorem) Suppose that V is a vector space and that $U \subset V$ is a subset of V. Show that

$$\mu u_1 + \nu u_2 \in U \text{ for all } u_1, u_2 \in U, \mu, \nu \in \mathbb{R}$$

implies that U is a subspace of V. (In other words, check all the vector space requirements for U.)

2. Let $P_3^{\mathbb{R}}$ be the vector space of polynomials of degree 3 or less in the variable x. Check whether

$$x - x^3 \in \operatorname{span}\{x^2, 2x + x^2, x + x^3\}$$

3. Let U and W be subspaces of V. Are:

   (a) $U \cup W$

   (b) $U \cap W$

   also subspaces? Explain why or why not. Draw examples in $\mathbb{R}^3$.

Hint

16 Problems: Linear Independence

1. Let B^n be the space of $n \times 1$ bit-valued matrices (*i.e.*, column vectors) over the field $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Remember that this means that the coefficients in any linear combination can be only 0 or 1, with rules for adding and multiplying coefficients given here.

   (a) How many different vectors are there in B^n?

   (b) Find a collection S of vectors that span B^3 and are linearly independent. In other words, find a basis of B^3.

   (c) Write each other vector in B^3 as a linear combination of the vectors in the set S that you chose.

   (d) Would it be possible to span B^3 with only two vectors?

2. Let e_i be the vector in $\mathbb{R}^n$ with a 1 in the ith position and 0's in every other position. Let v be an arbitrary vector in $\mathbb{R}^n$.

   (a) Show that the collection $\{e_1, \ldots, e_n\}$ is linearly independent.

   (b) Demonstrate that $v = \sum_{i=1}^{n}(v \bullet e_i)e_i$.

   (c) The $\operatorname{span}\{e_1, \ldots, e_n\}$ is the same as what vector space?

17 Problems: Basis and Dimension

1. (a) Draw the collection of all unit vectors in $\mathbb{R}^2$.

   (b) Let $S_x = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x \right\}$, where x is a unit vector in $\mathbb{R}^2$. For which x is S_x a basis of $\mathbb{R}^2$?

2. Let B^n be the vector space of column vectors with bit entries $0, 1$. Write down every basis for B^1 and B^2. How many bases are there for B^3? B^4? Can you make a conjecture for the number of bases for B^n?

   (Hint: You can build up a basis for B^n by choosing one vector at a time, such that the vector you choose is not in the span of the previous vectors you've chosen. How many vectors are in the span of any one vector? Any two vectors? How many vectors are in the span of any k vectors, for $k \leq n$?)

3. Suppose that V is an n-dimensional vector space.

   (a) Show that any n linearly independent vectors in V form a basis.

   (Hint: Let $\{w_1, \ldots, w_m\}$ be a collection of n linearly independent vectors in V, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Apply the method of Lemma 19.2 to these two sets of vectors.)

   (b) Show that any set of n vectors in V which span V forms a basis for V.

   (Hint: Suppose that you have a set of n vectors which span V but do not form a basis. What must be true about them? How could you get a basis from this set? Use Corollary 19.3 to derive a contradiction.)

4. Let S be a collection of vectors $\{v_1, \ldots, v_n\}$ in a vector space V. Show that if every vector w in V can be expressed uniquely as a linear combination of vectors in S, then S is a basis of V. In other words: suppose that for every vector w in V, there is exactly one set of constants $c^1, \ldots, c^n$ so that $c^1 v_1 + \cdots + c^n v_n = w$. Show that this means that the set S is linearly independent and spans V. (This is the converse to the theorem in the lecture.)

5. Vectors are objects that you can add together; show that the set of all linear transformations mapping $\mathbb{R}^3 \to \mathbb{R}$ is itself a vector space. Find a basis for this vector space. Do you think your proof could be modified to work for linear transformations $\mathbb{R}^n \to \mathbb{R}$?

   (Hint: Represent $\mathbb{R}^3$ as column vectors, and argue that a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}$ is just a row vector.)

   (Hint: If you are stuck or just curious, look up "dual space.")

18 Problems: Eigenvalues and Eigenvectors

1. Let $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find all eigenvalues of M. Does M have two independent[3] eigenvectors? Can M be diagonalized?

[3] Independence of vectors will be explained later; for now, think "not parallel".

2. Consider $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $L(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

(a) Write the matrix of L in the basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$.

(b) When $\theta \neq 0$, explain how L acts on the plane. Draw a picture.

(c) Do you expect L to have invariant directions?

(d) Try to find real eigenvalues for L by solving the equation

$$L(v) = \lambda v.$$

(e) Are there complex eigenvalues for L, assuming that $i = \sqrt{-1}$ exists?

3. Let L be the linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x,y,z) = (x+y, x+z, y+z)$. Let e_i be the vector with a one in the ith position and zeros in all other positions.

   (a) Find Le_i for each i.

   (b) Given a matrix $M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_3 \end{pmatrix}$, what can you say about Me_i for each i?

   (c) Find a 3×3 matrix M representing L. Choose three nonzero vectors pointing in different directions and show that $Mv = Lv$ for each of your choices.

   (d) Find the eigenvectors and eigenvalues of M.

19 Problems: Eigenvalues and Eigenvectors II

1. Explain why the characteristic polynomial of an $n \times n$ matrix has degree n. Make your explanation easy to read by starting with some simple examples, and then use properties of the determinant to give a *general* explanation.

2. Compute the characteristic polynomial $P_M(\lambda)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, since we can evaluate polynomials on square matrices, we can plug M into its characteristic polynomial and find the *matrix* $P_M(M)$. What do you find from this computation? Does something similar hold for 3×3 matrices? What about $n \times n$ matrices?

3. *Discrete dynamical system.* Let M be the matrix given by

$$M = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Given any vector $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, we can create an infinite sequence of vectors $v(1), v(2), v(3)$, and so on using the rule

$$v(t+1) = Mv(t) \text{ for all natural numbers } t.$$

(This is known as a *discrete dynamical system* whose *initial condition* is $v(0)$.)

(a) Find all eigenvectors and eigenvalues of M.

(b) Find all vectors $v(0)$ such that

$$v(0) = v(1) = v(2) = v(3) = \cdots$$

(Such a vector is known as a *fixed point* of the dynamical system.)

(c) Find all vectors $v(0)$ such that $v(0), v(1), v(2), v(3), \ldots$ all point in the same direction. (Any such vector describes an *invariant curve* of the dynamical system.)

20 Problems: Diagonalization

1. Let $P_n(t)$ be the vector space of polynomials of degree n or less, and $\frac{d}{dt} : P_n(t) \mapsto P_{n-1}(t)$ be the derivative operator. Find the matrix of $\frac{d}{dt}$ in the bases $\{1, t, \ldots, t^n\}$ for $P_n(t)$ and $\{1, t, \ldots, t^{n-1}\}$ for $P_{n-1}(t)$.

2. When writing a matrix for a linear transformation, we have seen that the choice of basis matters. In fact, even the order of the basis matters!

   - Write all possible reorderings of the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$.
   - Write each change of basis matrix between the standard basis $\{e_1, e_2, e_3\}$ and each of its reorderings. Make as many observations as you can about these matrices: what are their entries? Do you notice anything about how many of each type of entry appears in each row and column? What are their determinants? (Note: These matrices are known as *permutation matrices*.)
   - Given the linear transformation $L(x, y, z) = (2y - z, 3x, 2z + x + y)$, write the matrix M for L in the standard basis, and two other reorderings of the standard basis. How are these matrices related?

3. When is the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ diagonalizable? Include examples in your answer.

4. Show that similarity of matrices is an *equivalence relation*. (The definition of an equivalence relation is given in Homework 0.)

5. *Jordan form*

- Can the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
- Can the matrix $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
- Can the $n \times n$ matrix $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.

  Note: It turns out that every complex matrix is similar to a block matrix whose diagonal blocks look like diagonal matrices or the ones above and whose off-diagonal blocks are all zero. This is called the *Jordan form* of the matrix.

21 Problems: Orthonormal Bases

1. Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

   (a) Write D in terms of the vectors e_1 and e_2, and their transposes.

   (b) Suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Show that D is similar to

   $$M = \frac{1}{ad - bc} \begin{pmatrix} \lambda_1 ad - \lambda_2 bc & -(\lambda_1 - \lambda_2)ab \\ (\lambda_1 - \lambda_2)cd & -\lambda_1 bc + \lambda_2 ad \end{pmatrix}.$$

   (c) Suppose the vectors $\begin{pmatrix} a & b \end{pmatrix}$ and $\begin{pmatrix} c & d \end{pmatrix}$ are orthogonal. What can you say about M in this case? (Hint: think about what M^T is equal to.)

2. Suppose $S = \{v_1, \dots, v_n\}$ is an *orthogonal* (not orthonormal) basis for $\mathbb{R}^n$. Then we can write any vector v as $v = \sum_i c^i v_i$ for some constants c^i. Find a formula for the constants c^i in terms of v and the vectors in S.

3. Let u, v be independent vectors in $\mathbb{R}^3$, and $P = \text{span}\{u, v\}$ be the plane spanned by u and v.

   (a) Is the vector $v^{\perp} = v - \frac{u \cdot v}{u \cdot u} u$ in the plane P?

   (b) What is the angle between $v^{\perp}$ and u?

   (c) Given your solution to the above, how can you find a third vector perpendicular to both u and $v^{\perp}$?

   (d) Construct an orthonormal basis for $\mathbb{R}^3$ from u and v.

   (e) Test your abstract formulae starting with

$$u = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}.$$

22 Problems: Gram-Schmidt and Orthogonal Complements

1. Find the QR factorization of
$$M = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}.$$

2. Suppose u and v are linearly independent. Show that u and $v^\perp$ are also linearly independent. Explain why $\{u, v^\perp\}$ are a basis for $\text{span}\{u, v\}$.

3. Repeat the previous problem, but with three independent vectors u, v, w, and $v^{\perp}$ and $w^{\perp}$ as defined in the lecture.

4. Given any three vectors u, v, w, when do $v^{\perp}$ or $w^{\perp}$ vanish?

5. For U a subspace of W, use the subspace theorem to check that $U^\perp$ is a subspace of W.

6. This question will answer the question, "If I choose a bit vector *at random*, what is the probability that it lies in the span of some other vectors?"

   i. Given a collection S of k bit vectors in B^3, consider the bit matrix M whose columns are the vectors in S. Show that S is linearly independent if and only if the kernel of M is trivial.

   ii. Give some method for choosing a random bit vector v in B^3. Suppose S is a collection of 2 linearly independent bit vectors in B^3. How can we tell whether $S \cup \{v\}$ is linearly independent? Do you think it is likely or unlikely that $S \cup \{v\}$ is linearly independent? Explain your reasoning.

   iii. If P is the characteristic polynomial of a 3×3 bit matrix, what must the degree of P be? Given that each coefficient must be either 0 or 1, how many possibilities are there for P? How many of these possible characteristic polynomials have 0 as a root? If M is a 3×3 bit matrix chosen at random, what is the probability that it has 0 as an eigenvalue? (Assume that you are choosing a random matrix M in such a way as to make each characteristic polynomial equally likely.) What is the probability that the columns of M form a basis for B^3? (Hint: what is the relationship between the kernel of M and its eigenvalues?)

   Note: We could ask the same question for real vectors: If I choose a real vector at random, what is the probability that it lies in the span of some other vectors? In fact, once we write down a reasonable way of choosing a random real vector, if I choose a real vector in $\mathbb{R}^n$ at random, the probability that it lies in the span of $n-1$ other real vectors is 0!

23 Problems: Diagonalizing Symmetric Matrices

1. (On Reality of Eigenvectors)

   (a) Suppose $z = x + iy$ where $x, y \in \mathbb{R}, i = \sqrt{-1}$, and $\overline{z} = x - iy$. Compute $z\overline{z}$ and $\overline{z}z$ in terms of x and y. What kind of numbers are $z\overline{z}$ and $\overline{z}z$? (The complex number $\overline{z}$ is called the *complex conjugate* of z).

   (b) Suppose that $\lambda = x + iy$ is a complex number with $x, y \in \mathbb{R}$, and that $\lambda = \overline{\lambda}$. Does this determine the value of x or y? What kind of number must λ be?

   (c) Let $x = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} \in \mathbb{C}^n$. Let $x^\dagger = \begin{pmatrix} \overline{z^1} & \cdots & \overline{z^n} \end{pmatrix} \in \mathbb{C}^n$. Compute $x^\dagger x$. Using the result of part 1a, what can you say about the number $x^\dagger x$? (E.g., is it real, imaginary, positive, negative, etc.)

   (d) Suppose $M = M^T$ is an $n \times n$ symmetric matrix with real entries. Let λ be an eigenvalue of M with eigenvector x, so $Mx = \lambda x$. Compute:
   $$\frac{x^\dagger M x}{x^\dagger x}$$

   (e) Suppose Λ is a 1×1 matrix. What is Λ^T?

   (f) What is the size of the matrix $x^\dagger M x$?

   (g) For any matrix (or vector) N, we can compute $\overline{N}$ by applying complex conjugation to each entry of N. Compute $\overline{(x^\dagger)^T}$. Then compute $\overline{(x^\dagger M x)^T}$.

   (h) Show that $\lambda = \overline{\lambda}$. Using the result of a previous part of this problem, what does this say about λ?

2. Let $x_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $a^2 + b^2 + c^2 = 1$. Find vectors x_2 and x_3 such that $\{x_1, x_2, x_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

3. (Dimensions of Eigenspaces)

(a) Let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$. Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A. What is the sum of the dimensions of the eigenspaces of A?

(c) Based on your answer to the previous part, guess a formula for the sum of the dimensions of the eigenspaces of a real $n \times n$ symmetric matrix. Explain why your formula must work for any real $n \times n$ symmetric matrix.

24 Problems: Kernel, Range, Nullity, Rank

1. Let $L\colon V \to W$ be a linear transformation. Show that $\ker L = \{0_V\}$ if and only if L is one-to-one:

   (a) First, suppose that $\ker L = \{0_V\}$. Show that L is one-to-one. Think about methods of proof–does a proof by contradiction, a proof by induction, or a direct proof seem most appropriate?

   (b) Now, suppose that L is one-to-one. Show that $\ker L = \{0_V\}$. That is, show that 0_V is in $\ker L$, and then show that there are no other vectors in $\ker L$.

2. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Explain why

$$L(V) = \operatorname{span}\{L(v_1), \ldots, L(v_n)\}.$$

3. Suppose $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ whose matrix M in the standard basis is row equivalent to the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Explain why the first three columns of the original matrix M form a basis for $L(\mathbb{R}^4)$.
Find and describe and algorithm (*i.e.* a general procedure) for finding a basis for $L(\mathbb{R}^n)$ when $L\colon \mathbb{R}^n \to \mathbb{R}^m$.
Finally, use your algorithm to find a basis for $L(\mathbb{R}^4)$ when $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation whose matrix M in the standard basis is

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 4 & 1 & 1 & 6 \end{pmatrix}.$$

4. Claim: If $\{v_1, \ldots, v_n\}$ is a basis for $\ker L$, where $L \colon V \to W$, then it is always possible to extend this set to a basis for V.

   Choose a simple yet non-trivial linear transformation with a non-trivial kernel and verify the above claim for the transformation you choose.

5. Let $P_n(x)$ be the space of polynomials in x of degree less than or equal to n, and consider the derivative operator $\frac{\partial}{\partial x}$. Find the dimension of the kernel and image of $\frac{\partial}{\partial x}$.

   Now, consider $P_2(x, y)$, the space of polynomials of degree two or less in x and y. (Recall that xy is degree two, y is degree one and x^2y is degree three, for example.) Let $L = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. (For example, $L(xy) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy) = y + x$.) Find a basis for the kernel of L. Verify the dimension formula in this case.

25 Problems: Least Squares

1. Let $L : U \to V$ be a linear transformation. Suppose $v \in L(U)$ and you have found a vector $u_{\rm ps}$ that obeys $L(u_{\rm ps}) = v$.

   Explain why you need to compute $\ker L$ to describe the solution space of the linear system $L(u) = v$.

2. Suppose that M is an $m \times n$ matrix with trivial kernel. Show that for any vectors u and v in $\mathbb{R}^m$:

   - $u^T M^T M v = v^T M^T M u$
   - $v^T M^T M v \geq 0$.
   - If $v^T M^T M v = 0$, then $v = 0$.

   (Hint: Think about the dot product in $\mathbb{R}^n$.)