Math 21B **Midterm 1** **Fall 2010**

1. Evaluate the following indefinite integrals.

   (a)

$$\int \sin x \cos x \, dx$$

$$\left[\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right.$$

$$\begin{aligned} &= \int u du \\ &= \tfrac{1}{2}u^2 + C \\ &= \tfrac{1}{2}\sin^2 x + C \end{aligned}$$

   (b)

$$\int \frac{x-1}{x+1} \, dx$$

$$\left[\begin{array}{l} u = x+1 \\ du = dx \end{array} \right.$$

$$\begin{aligned} &= \int \frac{u-2}{u} \, du \\ &= \int 1 - \frac{2}{u} \, du \\ &= u - 2\ln|u| + C \\ &= x + 1 - 2\ln|x+1| + C \end{aligned}$$

2. Evaluate the following definite integrals.

   (a)

$$\int_0^1 \sqrt{1+2r} \, dr$$

$$\left[\begin{array}{l} u = 1+2r \\ du = 2dr \end{array} \right.$$

$$\begin{aligned} &= \tfrac{1}{2}\int_1^3 \sqrt{u} \, du \\ &= \tfrac{1}{3}u^{3/2}]_1^5 \\ &= \tfrac{1}{3}(3^{3/2} - 1) \end{aligned}$$

   (b)

$$\int_{-1}^1 1 - \sqrt{1-x^2} \, dx$$

From geometric considerations, the area under the graph is $2 - \pi/2$.

3.

$$\int_{-1}^{4} |x| \, dx$$

From geometric considerations, the area under the graph is $\frac{17}{2}$.

4.

$$\int \frac{1}{e^x + e^{-x} + 2} \, dx$$

$$\left[\begin{array}{l} u = e^x \\ du = e^x dx = u dx \end{array} \right.$$

$$\begin{aligned} &= \int \frac{1}{u + 1/u + 2} \left(\frac{1}{u} \right) \, du \\ &= \int \frac{1}{u^2 + 2u + 1} \, du \\ &= \int \frac{1}{(u+1)^2} \, du \\ &= -(u+1)^{-1} + C \\ &= -(e^x + 1)^{-1} + C \end{aligned}$$

5. For x with $0 \leq x \leq 1$, define $g(x) = \int_0^{\pi} t(\sin t)^x \, dt$. Find the maximum value of g over the interval $[0, 1]$.

Solution. Suppose that x is a number in $[0, 1]$. Since $0 \leq \sin t \leq 1$ for all $t \in [0, \pi]$, we have

$$(\sin t)^x \leq 1 = (\sin t)^0$$

It follows that

$$t(\sin t)^x \leq t(\sin t)^0$$

for all t in $[0, \pi]$. Thus by the domination property (see property 7 in table 5.3 and also problem 10 from homework 2), we have

$$\int_o^{\pi} t(\sin t)^x \, dt \leq \int_0^{\pi} t(\sin t)^0 \, dt,$$

i.e., $g(x) \leq g(0)$. It follows that the max occurs at $g(0) = \int_0^{\pi} t \, dt = \frac{1}{2}\pi^2$.

6. Find $\lim_{x \to 0} \frac{1}{x} \int_0^x \cos(t^2) \, dt$.

Solution. Let $A(x) = \int_0^x \cos(t^2) \, dt$. Then by the Fundamental Theorem of Calculus, we have $A'(x) = \cos(x^2)$. Hence

$$\begin{aligned} \lim_{x \to 0} \frac{A(x)}{x} &= \lim_{x \to 0} \frac{A(x) - A(0)}{x} \\ &= A'(0) \\ &= \cos(0^2) = 1 \end{aligned}$$

One could also use l'Hopital's rule here (see the solution to problem 6 from homework 2).

7. The gas mileage of a car depends on its velocity. When the velocity is v, the gas mileage is $f(v) = e^{-v^2}$ miles/gallon. If the velocity at time t is $v(t) = t$ miles/hour, how many gallons of gas are used after 5 hours?

Solution. Let

$$g(t) = \text{gas used by time } t;$$

$$s(t) = \text{position at time } t.$$

From the definition of velocity,

$$\frac{ds}{dt} = v(t) = t.$$

From the equation given for gas mileage,

$$\frac{ds}{dg} = e^{-v^2},$$

which implies that

$$\frac{dg}{ds} = e^{v^2} = e^{t^2}.$$

Hence by the chain rule

$$\begin{aligned} \frac{dg}{dt} &= \frac{dg}{ds}\frac{ds}{dt} \\ &= e^{t^2} \cdot t. \end{aligned}$$

The amount of gas used is

$$\begin{aligned} \int_0^5 g'(t)\,dt &= \int_0^5 te^{t^2}\,dt \\ &= \tfrac{1}{2}e^{t^2}]_0^5 \\ &= \tfrac{1}{2}e^{25} - \tfrac{1}{2} \quad \text{gallons.} \end{aligned}$$

Although the calculations here are not difficult, this problem is more conceptual than the others and a majority of the class got 0 points. This was the only problem on the exam not inspired by a homework problem.