Math 21B **Midterm 2 Solutions** **Fall 2010**

1. (20 points.) The region bounded by the graphs of $y = 2\sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the x-axis. Find the volume of the resulting solid.

   Solution. Find intersection point:

$$\begin{aligned} 2 &= 2\sqrt{x} \\ \sqrt{x} &= 1 \\ x &= 1 \end{aligned}$$

   Washer:

$$\begin{aligned} V &= \pi \int_0^1 2^2 - \left(2\sqrt{x}\right)^2 \; dx \\ &= \pi \int_0^1 4 - 4x \; dx \\ &= \pi \Big[4x - 2x^2\Big]_0^1 = 2\pi. \end{aligned}$$

2. (20 points.) The region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2/8$ is revolved about the y-axis. Find the volume of the resulting solid.

   Solution. Find intersection points:

$$\begin{aligned} \sqrt{x} &= x^2/8 \\ 8\sqrt{x} &= x^2 \\ 64x &= x^4 \\ 0 &= x^4 - 64x = x(x^3 - 64) \\ \\ \Rightarrow x &= 0, 4 \end{aligned}$$

   Shell:

$$\begin{aligned} V &= 2\pi \int_0^4 x(\sqrt{x} - \frac{x^2}{8}) \; dx \\ &= 2\pi \int_0^4 x^{3/2} - \frac{x^3}{8} \; dx \\ &= 2\pi \Big[\frac{2}{5}x^{5/2} - \frac{1}{32}x^4\Big]_0^4 \\ &= 2\pi[\frac{2}{5} \cdot 32 - 8] = \frac{48\pi}{5} \end{aligned}$$

3. (20 points.) Find the area of the region bounded by the graphs of $y = 0$, $y = 2$, $y = \sqrt{x}$ and $y = \sqrt{x - 1}$.

Solution. If you draw a picture you see that horizontal strips are better. The curve on the left is $x = y^2$ and the curve on the right is $x = y^2 + 1$. So

$$\begin{aligned} A &= \int_0^2 (y^2+1) - y^2 \, dy \\ &= \int_0^2 1 \, dy = 2 \end{aligned}$$

4. (20 points.) A car traveling at a constant speed of 10 miles per hour has a deteriorating engine. After t hours the gas mileage is $20/(t+1)$ miles per gallon. How far will the car go on 6 gallons of gas?

Solution. Let $s(t)$ be the position at time t. Then

$$\frac{ds}{dt} = 10; \qquad \frac{ds}{dg} = \frac{20}{t+1}.$$

Re-arranging terms gives

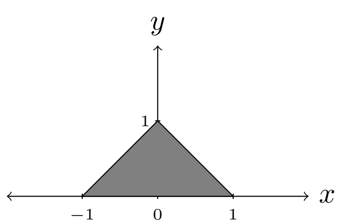
$$dg = \tfrac{1}{2}(t+1)dt$$

The gas used by time T is

$$\begin{aligned} \int dg &= \int_0^T \tfrac{1}{2}(t+1) \, dt \\ &= \tfrac{1}{4}(t+1)^2\Big]_0^T \\ &= \tfrac{1}{4}\Big[(T+1)^2 - 1\Big] \end{aligned}$$

This is 6 when $T = 4$. After 4 hours the car has gone $10 \times 4 = 40$ miles.

5. (20 points.) Find the mass of the triangular region below. All lengths are in meters, and the density of the region is given by $\delta(y) = y$ grams/m^2.



Solution. Consider a horizontal strip of width dy at height y. It has length $L(y) = 2(1-y)$, which can be seen by the following calculation: The line in the first quadrant is the graph of $y = 1 - x$, or $x = 1 - y$, and $L(y) = 2x = 2(1-y)$. Thus

$$\begin{aligned} \text{strip area} &= L(y)dy \\ \Rightarrow \text{strip mass } dm &= \delta(y)L(y)dy \\ &= y \cdot 2(1-y)dy \end{aligned}$$

Hence

$$
\begin{aligned}
m &= \int dm \\
&= \int_0^1 2y(1-y)\,dy \\
&= 2\int_0^1 y - y^2\,dy \\
&= 2\Big[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big]_0^1 = 1/3 \text{ grams}
\end{aligned}
$$

6. (20 points.) Find the area of the surface obtained by revolving the curve $y = x+1,\ 0 \le x \le 1$ about the y-axis.

   Solution. We have

$$
\begin{aligned}
ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \sqrt{1+1^2}\,dx \\
&= \sqrt{2}\,dx,
\end{aligned}
$$

   so

$$
\begin{aligned}
A &= \int_0^1 2\pi x ds \\
&= \int_0^1 2\pi x\sqrt{2}dx \\
&= 2\sqrt{2}\pi \int_0^1 xdx \\
&= 2\sqrt{2}\pi\Big[\tfrac{1}{2}x^2\Big]_0^1 = \sqrt{2}\pi.
\end{aligned}
$$

7. (20 points.) A rope 50 meters long weighing 2 Newtons per meter is hanging over the edge of a tall building and does not touch the ground. How much work is required to lift the entire rope to the top of the building?

   Solution. When the rope extends a distance x from the top of the building, the weight of the part hanging down is $F(x) = 2x$. Thus the work is

$$
\begin{aligned}
W &= \int_0^{50} F(x)\,dx \\
&= \int_0^{50} 2x\,dx \\
&= x^2\big]_0^{50} = 2500 \text{ Nm}
\end{aligned}
$$