A NOTE ON PROPERLY DISCONTINUOUS ACTIONS

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Abstract. We compare various notions of proper discontinuity for group actions. We also discuss fundamental domains and criteria for cocompactness.

To the memory of Sasha Anan'in

1. Introduction

This note is meant to clarify the relation between different commonly used definitions of proper discontinuity without the local compactness assumption for the underlying topological space. Much of the discussion applies to actions of nondiscrete locally compact Hausdorff topological groups, but, since my primary interest is geometric group theory, I will mostly work with discrete groups. All group actions are assumed to be continuous, in other words, for discrete groups, these are homomorphisms from abstract groups to groups of homeomorphisms of topological spaces. This combination of *continuous* and *properly discontinuous*, sadly, leads to the ugly terminology "a continuous properly discontinuous action." A better terminology might be that of a *properly discrete* action, since it refers to proper actions of discrete groups.

Throughout this note, I will be working only with topological spaces which are 1st countable, since spaces most common in metric geometry, geometric topology, algebraic topology and geometric group theory satisfy this property. One advantage of this assumption is that if (x_n) is a sequence converging to a point $x \in X$, then the subset $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, which is not true if we work with nets instead of sequences. However, I will try to avoid the local compactness assumption whenever possible, since many spaces appearing in metric geometry and geometric group theory (e.g. asymptotic cones) and algebraic topology (e.g. CW complexes) are not locally compact. (Recall that topological space X is *locally compact* if every point has a basis of topology consisting of relatively compact subsets.)

In the last three sections of the note I discuss several concepts related to properly discontinuous actions. In Section 5 I discuss cocompactness of group actions. In Section 6 I discuss group-invariant metrics. In particular, under suitable assumptions I will prove existence of an invariant complete geodesic metric (Theorem 25). In Section 7 I discuss fundamental sets and regions. The main result of this section is Theorem 61 which uses Voronoi tessellations to establish existence of fundamental regions and domains for free properly discontinuous actions on proper geodesic metric spaces.

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2. Group actions

A *topological group* is a group G equipped with a topology such that the multiplication and inversion maps

$$G \times G \to G, (g,h) \mapsto gh, G \to G, g \mapsto g^{-1}$$

are both continuous. A *discrete group* is a group with discrete topology. Every discrete group is clearly a topological group.

A *left continuous action* of a topological group G on a topological space X is a continuous map

$$\lambda : G \times X \to X$$

satisfying

1. $\lambda(1_G, x) = x$ for all $x \in X$.
2. $\lambda(gh, x) = \lambda(g, \lambda(h, x))$, for all $x \in X$, $g, h \in G$.

From this, it follows that the map $\rho : G \to Homeo(X)$

$$\rho(g)(x) = \lambda(g, x),$$

is a group homomorphism, where the group operation $\phi\psi$ on $Homeo(X)$ is the composition $\phi \circ \psi$.

If G is discrete, then every homomorphism $G \to Homeo(X)$ defines a left continuous action of G on X.

The shorthand for $\rho(g)(x)$ is gx or $g \cdot x$. Similarly, for a subset $A \subset X$, GA or $G \cdot A$, denotes the *orbit of* A under the G-action:

$$GA = \bigcup_{g \in G} gA.$$

The *quotient space* X/G (also frequently denoted $G \backslash X$), of X by the G-action, is the set of G-orbits of points in X, equipped with the quotient topology: The elements of X/G are equivalence classes in X, where $x \sim y$ when $Gx = Gy$ (equivalently, $y \in Gx$).

The *stabilizer* of a point $x \in X$ under the G-action is the subgroup $G_x < G$ given by

$$\{g \in G : gx = x\}.$$

An action of G on X is called *free* if $G_x = \{1\}$ for all $x \in X$. Assuming that X is Hausdorff, G_x is closed in G for every $x \in X$.

Example 1. *An example of a left action of G is the action of G on itself via left multiplication:*

$$\lambda(g, h) = gh.$$

In this case, the common notation for $\rho(g)$ is L_g. This action is free.

3. Proper maps

Properness of certain maps is the most common form of defining proper discontinuity; sadly, there are two competing notions of properness in the literature.

A continuous map $f : X \to Y$ of topological spaces is *proper in the sense of Bourbaki*, or simply *Bourbaki–proper* (cf. [6, Ch. I, §10, Theorem 1]) if f is a closed map (images of closed subsets are closed) and point–preimages $f^{-1}(y), y \in Y$, are compact. A continuous map $f : X \to Y$ is *proper* (and this is the most common definition) if for every compact subset $K \subset X$, $f^{-1}(K)$ is compact.

It is noted in [6, Ch. I, §10; Prop. 7] that if X is Hausdorff and Y is locally compact, then f is Bourbaki–proper if and only if f is proper.

The advantage of the notion of Bourbaki-properness is that it applies in the case of Zariski topology, where spaces tend to be compact[1] (every subset of a finite-dimensional affine space is Zariski-compact) and, hence, the standard notion of properness is useless.

Since our goal is to trade local compactness for 1st countability, I will prove a lemma which appears as a corollary in [20]:

Lemma 2. *If $f : X \to Y$ is proper, and X, Y are Hausdorff and 1st countable, then f is Bourbaki-proper.*

Proof. We only have to verify that f is closed. Suppose that $A \subset X$ is a closed subset. Since Y is 1st countable, it suffices to show that for each sequence (x_n) in A such that $(f(x_n))$ converges to $y \in Y$, there is a subsequence (x_{n_k}) which converges to some $x \in A$ such that $f(x) = y$. The subset $C = \{y\} \cup \{f(x_n) : n \in \mathbb{N}\} \subset Y$ is compact. Hence, by properness of f, $K = f^{-1}(C)$ is also compact. Since X is Hausdorff, and K is compact, follows that (x_n) subconverges to a point $x \in K$. By continuity of f, $f(x) = y$. Since A is closed, $x \in A$. □

Remark 3. *This lemma still holds if one were to replace the assumption that X is 1st countable by surjectivity of f, see* [20].

The converse (each Bourbaki–proper map is proper) is proven in [6, Ch. I, §10; Prop. 6] without any restrictions on X, Y. Hence:

Corollary 4. *For maps between 1st countable Hausdorff spaces, Bourbaki-properness is equivalent to properness.*

4. Proper discontinuity

Suppose that X is a 1st countable Hausdorff topological space, G a discrete group and $G \times X \to X$ a (continuous) action. I use the notation $g_n \to \infty$ in G to indicate that g_n converges to ∞ in the 1-point compactification $G \cup \{\infty\}$ of G, i.e. for every finite subset $F \subset G$,

$$\mathrm{card}(\{n : g_n \in F\}) < \infty.$$

Given a group action $G \times X \to X$ and two subsets $A, B \subset X$, the *transporter subset* $(A|B)_G$ is defined as

$$(A|B)_G := \{g \in G : gA \cap B \neq \emptyset\}.$$

Properness of group actions is (typically) stated using certain transporter sets.

Definition 5. *Two points $x, y \in X$ are said to be G-*dynamically related *if there is a sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ in X such that $g_n x_n \to y$.*

A point $x \in X$ is said to be a *wandering point* of the G-action if there is a neighborhood U of x such that $(U|U)_G$ is finite.

Lemma 6. *Suppose that the action $G \times X \to X$ is wandering at a point $x \in X$. Then the G-action has a G-*slice *at x, i.e. a neighborhood $W_x \subset U$ which is G_x-stable and for all $g \notin G_x$, $gW_x \cap W_x = \emptyset$.*

[1] quasicompact in the Bourbaki terminology

Proof. For each $g \in (U|U)_G - G_x$ we pick a neighborhood $V_g \subset U$ of x such that $gV_g \cap V_g = \emptyset$. Then the intersection

$$V := \bigcap_{g \in (U|U)_G - G_x} V_g$$

satisfies the property that $(V|V)_G = G_x$. Lastly, take

$$W_x := \bigcap_{g \in G_x} V. \quad \square$$

The next lemma is clear:

Lemma 7. *Assuming that X is Hausdorff and 1st countable, the action $G \times X \to X$ is wandering at x if and only if x is not dynamically related to itself.*

Given a group action $\alpha : G \times X \to X$, we have the natural map

$$\hat{\alpha} := \alpha \times \mathrm{id}_X : G \times X \to X \times X$$

where $\mathrm{id}_X : (g, x) \mapsto x$.

Definition 8. *An action α of a discrete group G on a topological space X is* Bourbaki–proper *if the map $\hat{\alpha}$ is Bourbaki-proper.*

Lemma 9. *If the action $\alpha : G \times X \to X$ of a discrete group G on a Hausdorff topological space X is Bourbaki-proper, then the quotient space X/G is Hausdorff.*

Proof. The quotient map $X \to X/G$ is an open map by the definition of the quotient topology on X/G. Since α is Bourbaki-proper, the image of the map $\hat{\alpha}$ is closed in $X \times X$. This image is the equivalence relation on $X \times X$ which use used to form the quotient X/G. Now, Hausdorffness of X/G follows from [6, Proposition 8 in I.8.3]. $\square$

Definition 10. *An action α of a discrete group G on a topological space X is* proper *if the map $\hat{\alpha}$ is proper.*

Note that the equivalence of (1) and (5) in the following theorem is proven in [6, Ch. III, §4.4, Proposition 7] without any assumptions on X.

Theorem 11. *Assuming that X is Hausdorff and 1st countable, the following are equivalent:*

(1) *The action $\alpha : G \times X \to X$ is Bourbaki-proper.*
(2) *For every compact subset $K \subset X$,*

$$\mathrm{card}((K|K)_G) < \infty.$$

(3) *The action $\alpha : G \times X \to X$ is proper, i.e. the map $\hat{\alpha}$ is proper.*
(4) *For every compact subset $K \subset X$, there exists an open neighborhood U of K such that* $\mathrm{card}((U|U)_G) < \infty$.
(5) *For any pair of points $x, y \in X$ there is a pair of neighborhoods U_x, V_x (of x, y respectively) such that* $\mathrm{card}((U_x|V_y)_G)) < \infty$.
(6) *There are no G-dynamically related points in X.*
(7) *Assuming, that G is countable and X is completely metrizable*[2] *: The G-stabilizer of every $x \in X$ is finite and for any two points $x \in X, y \in X - Gx$, there exists a pair of neighborhoods U_x, V_y (of x, resp. y) such that $\forall g \in G$, $gU_x \cap V_y = \emptyset$.*

[2]It suffices to assume that X is *hereditarily Baire*: Every closed subset of X is Baire.

(8) *Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous*[3]*: There is no $x \in X$ and a sequence $h_n \to \infty$ in G such that $h_n x \to x$.*
(9) *Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous: Every $x \in X$ is a wandering point of the G-action.*
(10) *Assuming that X is a CW complex and the action $G \times X \to X$ is cellular: Every point of X is wandering.*
(11) *Assuming that X is a CW complex the action $G \times X \to X$ is cellular: Every cell in X has finite G-stabilizer.*

Proof. The action α is Bourbaki-proper if and only if the map $\hat{\alpha}$ is proper (see Corollary 4) which is equivalent to the statement that for each compact $K \subset X$, the subset $(K|K)_G \times K$ is compact. Hence, (1) $\iff$ (2).

Assume that (3) holds, i.e. α is proper, equivalently, the map $\hat{\alpha}$ is proper. This means that for each compact $K \subset X$, $\hat{\alpha}^{-1}(K \times K) = \{(g, x) \in G \times K : x \in K, gx \in K\}$ is compact. This subset is closed in $G \times X$ and projects onto $(K|K)_G$ in the first factor and to the subset

$$(\star) \qquad \bigcup_{g \in (K|K)_G} g^{-1}(K)$$

in the second factor. Hence, properness of the action α implies finiteness of $(K|K)_G$, i.e. (2). Conversely, if $(K|K)_G$ is finite, compactness of $g^{-1}(K)$ for every $g \in G$ implies compactness of the union $(\star)$. Thus, (2) $\iff$ (3).

In order to show that (2)$\Rightarrow$(6), suppose that x, y are G-dynamically related points: There exists a sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ such that $g_n(x_n) \to y$. The subset

$$K = \{x, y\} \cup \{x_n, g_n(x_n) : n \in \mathbb{N}\}$$

is compact. However, $y_n \in g_n(K) \cap K$ for every n. A contradiction.

(6)$\Rightarrow$(5): Suppose that the neighborhoods U_x, V_y do not exist. Let $\{U_n\}_{n \in \mathbb{N}}$, $\{V_n\}_{n \in \mathbb{N}}$ be countable bases at x, y respectively. Then for every n there exists $g_n \in G$, such that $g_n(U_n) \cap V_n \neq \emptyset$ for infinitely many g_n's in G. After extraction, $g_n \to \infty$ in G. This yields points $x_n \in U_n, y_n = g_n(x_n) \in V_n$. Hence, $x_n \to x, y_n \to y$. Thus, x is G-dynamically related to y. A contradiction.

(5)$\Rightarrow$(4). Consider a compact $K \subset X$. Then for each $x \in K, y \in K$ there exist neighborhoods U_x, V_y such that $(U_x|V_y)_G$ is finite. The product sets $U_x \times V_y, x, y \in K$ constitute an open cover of K^2. By compactness of K^2, there exist $x_1, ..., x_n, y_1, ..., y_m \in K$ such that

$$K \subset U_{x_1} \cup ... \cup U_{x_n}$$
$$K \subset V_{y_1} \cup ... \cup V_{y_m}$$

and for each pair (x_i, y_j),

$$\text{card}(\{g \in G : gU_{x_i} \cap V_{y_j} \neq \emptyset\}) < \infty.$$

Setting

$$W := \bigcup_{i=1}^{n} U_{x_i}, V := \bigcup_{j=1}^{m} V_{y_j},$$

we see that

$$\text{card}((W|V)_G) < \infty.$$

Taking $U := V \cap W$ yields the required subset U.

The implication (4)$\Rightarrow$(2) is immediate.

[3]E.g. an isometric action.

This concludes the proof of equivalence of the properties (1)—(6).

(5)⇒(7): Finiteness of G-stabilizers of points in X is clear. Let x, y be points in distinct G-orbits. Let U'_x, V'_y be neighborhoods of x, y such that $(U'_x|V'_y)_G = \{g_1, ..., g_n\}$. For each i, since X is Hausdorff, there are disjoint neighborhoods V_i of y and W_i of $g_i(x_i)$. Now set

$$V_y := \bigcap_{i=1}^{n} V_i, \quad U_x := \bigcap_{i=1}^{n} g_i^{-1}(W_i).$$

Then $gU_x \cap V_y = \emptyset$ for every $g \in G$.

(7)⇒(6): It is clear that (7) implies that there are no dynamically related points with distinct G-orbits. In particular, every G-orbit in X is closed.

Assume now that X is completely metrizable and G is countable. Suppose that a point $x \in X$ is G-dynamically related to itself. Since the stabilizer G_x is finite, the point x is an accumulation point of Gx; moreover, Gx is closed in X. Hence, Gx is a closed perfect subset of X. Since X admits a complete metric, so does its closed subset Gx. Thus, for each $g \in G$, the complement $U_g := Gx - \{gx\}$ is open and dense in Gx. By the Baire Category Theorem, the countable intersection

$$\bigcap_{g \in G} U_g$$

is dense in Gx. However, this intersection is empty. A contradiction.

It is clear that (6)⇒(8) (without any extra assumptions).

(8)⇒(6). Suppose that X is a metric space and the G-action is equicontinuous. Equicontinuity implies that for each $z \in X$, a sequence $z_n \to z$ and $g_n \in G$,

$$g_n z_n \to gz.$$

Suppose that there exist a pair of G-dynamically related points $x, y \in X$: $\exists x_n \to x, g_n \in G$, $g_n x_n \to y$. By the equicontinuity of the action, $g_n x \to y$. Since $g_n \to \infty$, there exist subsequences $g_{n_i} \to \infty$ and $g_{m_i} \to \infty$ such that the products $h_i := g_{n_i}^{-1} g_{m_i}$ are all distinct. Then, by the equicontinuity,

$$h_i x \to x.$$

A contradiction.

The implications (5)⇒(9)⇒(8) and (5)⇒(10)⇒(11) are clear.

Lastly, let us prove the implication (11)⇒(2). We first observe that every CW complex is Hausdorff and 1st countable. Furthermore, every compact $K \subset X$ intersects only finitely many open cells e_λ in X. (Otherwise, picking one point from each nonempty intersection $K \cap e_\lambda$ we obtain an infinite closed discrete subset of K.) Thus, there exists a finite subset $E := \{e_\lambda : \lambda \in \Lambda\}$ of open cells in X such that for every $g \in (K|K)_G$, $gE \cap E \neq \emptyset$. Now, finiteness of $(K|K)_G$ follows from finiteness of cell-stabilizers in G. □

Unfortunately, the property that every point of X is a wandering point is frequently taken as the definition of proper discontinuity for G-actions, see e.g. [13, 18]. Items (8) and (10) in the above theorem provide a (weak) justification for this abuse of terminology. I feel that the better name for such actions is *wandering actions*.

Example 12. *Consider the action of $G = \mathbb{Z}$ on the punctured affine plane $X = \mathbb{R}^2 - \{(0,0)\}$, where the generator of $\mathbb{Z}$ acts via $(x, y) \mapsto (2x, \frac{1}{2}y)$. Then for any $p \in X$, the G-orbit Gp has no* *accumulation points in X. However, any two points $p = (x, 0), q = (0, y) \in X$ are dynamically related. Thus, the action of G is not proper.*

This example shows that the quotient space of a wandering action need not be Hausdorff.

Lemma 13. *Suppose that $G \times X \to X$ is a wandering action. Then each G-orbit is closed and discrete in X. In particular, the quotient space X/G is T1.*

Proof. Suppose that Gx accumulates at a point y. Then $Gx \cap W_y$ is nonempty, where W_y is a G-slice at y. It follows that all points of $Gx \cap W_y$ lie in the same W_y-orbit, which implies that $Gx \cap W_y = \{y\}$. □

There are several reasons to consider proper actions of discrete (and, more generally, locally compact) groups; one reason is that such each proper action of a discrete group yields an *orbi-covering map* in the case of smooth group actions on manifolds: $M \to M/G$ is an orbi-covering provided that the action of G on M is smooth (or, at least, locally smoothable). Another reason is that for a proper action on a Hausdorff space, $G \times X \to X$, the quotient X/G is again Hausdorff, see Lemma 9.

Question 14. *Suppose that G is a discrete group, $G \times X \to X$ is a free continuous action on an n-dimensional topological manifold X such that the quotient space X/G is a (Hausdorff) n-dimensional topological manifold. Does it follow that the G-action on X is proper?*

The answer to this question is negative if one merely assumes that X is a locally compact Hausdorff topological space and X/G is Hausdorff, see [10] (the action given there was even cocompact). Below is a different example. We begin by constructing a non-proper free continuous $\mathbb{R}$-action on a manifold, such that the quotient space is not just Hausdorff but is a manifold with boundary.

Example 15. *This is a variation on Example 12. We start with the space*

$$Z = \{(x, y) : x, y \in [0, \infty), (x, y) \neq (0, 0)\}.$$

Take the quotient space X of Z by the equivalence relation $(x, 0) \sim (0, \frac{1}{x})$. The space X is homeomorphic to the open Moebius band. The group $G = \mathbb{R}$ acts on Z continuously by

$$(t, (x, y)) \mapsto (2^t x, 2^{-t} y).$$

The above equivalence relation on X is preserved by the G-action and, hence, the G-action descends to a continuous G-action on X. It is easy to see that this action is free but not proper: The equivalence class of $(1, 0)$ is dynamically related to itself. Lastly, the quotient X/G is Hausdorff, homeomorphic to $[0, 1)$ (the equivalence class of $(1, 0)$ maps to $0 \in [0, 1)$).

Lastly, we use Example 15 to construct a non-proper free $\mathbb{Z}$-action with Hausdorff quotient. We continue with the notation of the previous example.

Example 16. *Let $Y \subset Z$ denote the following subset of Z (with the subspace topology):*

$$Y = \{(2^m, 0) : m \in \mathbb{Z}\} \cup \{(0, 2^n) : n \in \mathbb{Z}\} \cup \{(2^m, 2^n) : (m, n) \in \mathbb{Z}^2\}.$$

Let W denote the projection of Y to X. We take $\Gamma = \mathbb{Z} < G = \mathbb{R}$. This subgroup preserves Y and, hence, W. The quotient W/Γ is homeomorphic to $Y \cap \{(0, y) : y \in \mathbb{R}\}$, hence, is Hausdorff. At the same time, the Γ-action on W is non-proper.

5. Cocompactness

There are two common notions of cocompactness for group actions:

(1) $G \times X \to X$ is cocompact if there exists a compact $K \subset X$ such that $G \cdot K = X$.
(2) $G \times X \to X$ is cocompact if X/G is compact.

It is clear that (1)⇒(2), as the image of a compact under the continuous (quotient) map $p : X \to X/G$ is compact.

Lemma 17. *If X is locally compact then (2)⇒(1).*

Proof. For each $x \in X$ let U_x denote a relatively compact neighborhood of x in X. Then

$$V_x := p(U_x) = p(G \cdot U_x),$$

is compact since $G \cdot U_x$ is open in X. Thus, we obtain an open cover $\{V_x : x \in X\}$ of X/G. Since X/G is compact, this open cover contains a finite subcover

$$V_{x_1}, ..., V_{x_n}.$$

It follows that

$$p(\bigcup_{i=1}^{n} U_{x_i}) = X/G.$$

The set

$$K = \bigcup_{i=1}^{n} \overline{U_{x_i}}$$

is compact and $p(K) = X/G$. Hence, $G \cdot K = X$. □

Lemma 18. *Suppose that X is normal and Hausdorff, $G \times X \to X$ is a proper action of a discrete group, such that X/G is locally compact. Then X is locally compact.*

Proof. Pick $x \in X$. Let W_x be a slice for the G-action at x; then $W_x/G_x \to X/G$ is a topological embedding. Thus, our assumptions imply that W_x/G_x is compact for every $x \in X$. Let (x_α) be a net in W_x. Since W_x/G_x is compact, the net $(x_\alpha)/G$ contains a convergent subnet. Thus, after passing to a subnet, there exists $g \in G_x$ such that (gx_α) converges to some $x \in \overline{W_x}$. Hence, (x_α) subconverges to $g^{-1}(x)$. Thus, W_x is relatively compact. Since X is assumed to be normal, x admits a basis of relatively compact neighborhoods. □

Corollary 19. *For normal Hausdorff spaces X the two notions of cocompactness agree for proper discrete group actions on X.*

On the other hand, if the drop the properness condition, the two notions are not equivalent even for $\mathbb{Z}$-actions with Hausdorff quotients, see the example by R. de la Vega in [27].

6. Invariant metrics

We start with several general definitions. A discrete subset E of a metric space (X, d) will be called *metrically proper* if for some (equivalently, every) $p \in X$ the function

$$d(p, \cdot) : E \to \mathbb{R}_+$$

is proper. In other words, every metric ball contains only finitely many points of E. A *geodesic metric space*, is a metric space (X, d) where every two points x, y are connected by a geodesic segment, i.e. an isometric embedding $c : [a, b] \to (X, d)$ such that $c(a) = x, c(b) = y$. Geodesic segments connecting x to y need not be unique; however, one frequently denotes such segments xy by abusing the notation. We will also conflate geodesic segments and their images. Note that each locally compact complete geodesic metric space (X, d) is *proper*, i.e. closed metric balls in (X, d) are compact, see [8, Theorem 2.5.28].

An isometric action $G \times X \to X$ of a discrete group is *metrically proper* if G acts with finite point-stabilizers and one (equivalently, every) G-orbit in X is a metrically proper subset. In other words, for every $x \in X$ the function

$$g \mapsto d(x, gx)$$

is proper on G. This condition is stronger than properness of the action but is equivalent to properness of the G-action in the case of proper metric spaces (X, d). Given an isometric properly discontinuous G-action on X we define the function

$$\rho : X/G \to \mathbb{R}_+$$

sending each equivalence class $[x] \in X/G$ to

$$\inf\{d(gx, x) : g \in G \setminus \{1\}\}.$$

This function is 2-Lipschitz:

$$|\rho([x]) - \rho([y])| \leq 2d([x], [y]).$$

If the G-action is metrically proper, then the infimum in the definition of ρ is realized and if the action is also free then $\rho([x]) > 0$ for all $x \in X$. By abusing the notation, we will also denote this function $\rho(x)$.

Suppose that (X, d) is a metric space and G is a group acting isometrically and metrically properly on X. One defines the *quotient-metric* d_G on X/G by

$$d_G([x], [y]) = \min_{g \in G} d(x, Gy) = \min_{g,h \in G} d(gx, hy), \tag{20}$$

where $[x], [y] \in X/G$ are equivalence classes of points $x, y \in X$ under the equivalence relation defined by G. Then d_G is a metric on X/G which metrizes the quotient topology on X/G, see [21, Theorem 6.6.2]. By the construction, the quotient map $q : (X, d) \to (X/G, d_G)$ is 1-Lipschitz.

Lemma 21. *Suppose that the G-action on X is metrically proper. Then the following hold:*

1. *If the metric space (X, d) is geodesic and complete, then so is $(X/G, d_G)$.*
2. *If (X, d) is proper, so is $(X/G, d_G)$.*
3. *If the G-action is free then the quotient map $q : (X, d) \to (X/G, d_G)$ is a local isometry. More precisely, for every $x \in X$ the restriction of q to $B(x, \frac{1}{8}\rho(x))$ is an isometry onto $B([x], \frac{1}{8}\rho(x))$.*

Proof. 1a. Take points $[x], [y] \in X/G$. Pick their representatives $x, y \in X$ which realize the minimal distance between the corresponding G-orbits in X. Let $c : [0, T] \to xy \subset X$ be a geodesic connecting x to y. Then, by the definition of the metric d_G, the composition of c with the quotient map $q : X \to X/G$ is a geodesic in $(X/G, d_G)$ connecting $[x]$ to $[y]$.

1b. Suppose that (z_n) is a Cauchy sequence in X/G. Then the diameter D of the subset

$$\{z_n : n \in \mathbb{N}\} \subset X/G$$

is finite. We inductively choose a subsequence (z_{n_i}) in (z_n) such that

$$d_G(z_{n_i}, z_{n_{i+1}}) \leq \frac{D}{2^i}, i \in \mathbb{N}.$$

Concatenating geodesic segments $z_{n_i} z_{n_{i+1}}$ we obtain a piecewise-geodesic path

$$\gamma : [0, T) \to X$$

whose length T is at most

$$\sum_{n=1}^{\infty} \frac{D}{2^i} < \infty.$$

We then inductively lift each geodesic segment in γ to a geodesic segment in X and obtain a piecewise-geodesic path $c : [0,T) \to X$ of length T. Since (X,d) is complete, the path c extends continuously to T. Projecting $c(T)$ to X/G we obtain the limit of the subsequence (z_{n_i}). Hence, (z_n) converges as well.

2. Suppose that (X,d) is proper. Consider the closed metric ball $\bar{B}([x],R)$ in $(X/G,d_G)$. Then the closed ball $\bar{B}(x,R) \subset X$ projects onto $\bar{B}([x],R)$. Compactness of $\bar{B}(x,R)$ implies compactness of $\bar{B}([x],R)$.

3. Fix $x \in X$, set $R = \frac{1}{8}\rho(x)$ and consider points $y,z \in B(x,R)$. We have to verify that $d(y,z) = d(y,Gz)$. Take $g \in G \setminus \{1\}$. We have $|\rho(y) - \rho(x)| \le 2R$ and $|\rho(z) - \rho(x)| < 2R$ since ρ is 2-Lipschitz. Thus, $d(z,gz) > \rho(x) - 2R = \rho(x) - \frac{1}{4}\rho(x) = \frac{3}{4}\rho(x)$. By the triangle inequality,

$$d(y,gz) > \frac{3}{4}\rho[x) - 2R = \frac{3}{4}\rho(x) - \frac{1}{4}\rho(x) = \frac{1}{2}\rho(x) > 2R > d(y,z).$$

Lastly, for every $r > 0$ and $x \in X$, $q(B(x,r))$ is contained in $B([x],r)$ since the quotient map $q : (X,d) \to (X/G,d_G)$ is 1-Lipschitz. In the case $r = R$ as above, the fact that q restricts to an isometry on $B(x,R)$ implies the equality $q(B(x,R)) = B([x],R)$. □

It turns out that under some rather mild assumptions, given a proper action $G \times X \to X$, there is a G-invariant metric metrizing the topology on X:

Theorem 22. *Suppose that G is a locally compact Hausdorff group, X is locally compact, metrizable space, $G \times X \to X$ is a proper action and X/G is paracompact. Then X admits a G-invariant metric metrizing the topology on X*

See [17, Theorem 3]. Koszul also notes that if X is paracompact and locally connected, then X/G is paracompact. This theorem was improved in [1]:

Theorem 23. *Suppose that G is a locally compact Hausdorff group, X is locally compact, σ-compact metrizable space, and $G \times X \to X$ is a proper action. Then X admits a G-invariant proper metric metrizing the topology on X.*

A Riemannian version of these theorems holds in the context of smooth actions of Lie groups:

Theorem 24. *Suppose that X is a smooth manifold, G is a Lie group and $G \times X \to X$ is a smooth proper action. There there exists a G-invariant complete Riemannian metric on X.*

See [17, Theorem 2] for the existence of an invariant Riemannian metric and [14] for the existence of an invariant complete Riemannian metric.

We next discuss a construction of G-invariant *complete geodesic metrics* on more general topological spaces.

Theorem 25. *Suppose that X is a 2nd countable, connected and locally connected locally compact Hausdorff topological space. Suppose that $G \times X \to X$ is a proper action of a discrete countable group such that the fixed-point set of each nontrivial element of G is nowhere dense in X. Then X can be metrized using a G-invariant complete geodesic metric.*

Proof.

Lemma 26. *The quotient space $Y = X/G$ is locally compact, connected, locally connected and metrizable.*

Proof. Local compactness and connectedness of Y follows from that of X. The 2nd countability of X implies the 2nd countability of Y. By Lemma 9, Y is Hausdorff. Since Y is locally compact and Hausdorff, its one-point compactification is compact and Hausdorff, hence, regular. It follows that Y itself is regular. In view of the 2nd countability of Y, Urysohn's metrization theorem implies that Y is metrizable. $\square$

Remark 27. *Note that each locally compact metrizable space is also locally path-connected.*

It is proven in [26] that each locally compact, connected, locally connected metrizable space, such as Y, admits a complete geodesic metric d_Y which we fix from now on. Consider the projection $p : X \to Y$. According to [7, Theorem 6.2] (see also [2, Lemma 2]), the map p satisfies the path-lifting property: Given any path $c : [0,1] \to Y$, a point $x \in X$ satisfying $p(x) = c(0)$, there exists a path $\tilde{c} : [0,1] \to X$ such that $p \circ \tilde{c} = c$. (This result is, of course, much easier if the G-action is free, i.e. $p : X \to Y$ is a covering map.) We let $\mathcal{L}_X$ denote the set of paths in X which are lifts of rectifiable paths $c : [0,1] \to Y$. Clearly, the postcomposition of $\tilde{c} \in \mathcal{L}_X$ with an element of G is again in $\mathcal{L}_X$. Our next goal is to equip X with a G-invariant *length structure* using the family of paths $\mathcal{L}_X$. Such a structure is a function on $\mathcal{L}_X$ with values in $[0,\infty)$, satisfying certain axioms that can be found in [8, Section 2.1]. Verification of most of these axioms is straightforward, I will check only some (items 1, 2, 3 and 4 below).

1. If $\tilde{c} \in \mathcal{L}_X$ is a lift of a path c in Y, then we declare $\ell(\tilde{c})$ to be equal to the length of c.

2. If $\tilde{c}_i, i = 1, 2$, are paths in $\mathcal{L}_X$ (which are lifts of the paths c_1, c_2 respectively) whose concatenation $b = \tilde{c}_1 \star \tilde{c}_2$ is defined, then b is a lift of the concatenation $c_1 \star c_2$. Clearly, $\ell(b) = \ell(\tilde{c}_1) + \ell(\tilde{c}_2)$.

3. Let U be a neighborhood of some $x \in X$. We need to prove that

$$\inf_{\gamma}\{\ell(\gamma)\} > 0, \tag{28}$$

where the infimum is taken over all $\gamma = \tilde{c} \in \mathcal{L}_X$ connecting x to points of $X \setminus U$. It suffices to prove this claim in the case when U is G_x-invariant, satisfies

$$\overline{U} \cap g\overline{U} \neq \emptyset \iff g \in G_x, \tag{29}$$

and γ connects x to points of ∂U. Then $V = p(U)$ is a neighborhood of $y = p(x)$ in Y and the paths $c = p \circ \gamma$ connect y to points in ∂V. But the lengths of the paths c are clearly bounded away from zero and are equal to the lengths of their lifts $\tilde{c}$. Thus, we obtain the required bound (28).

4. Let us verify that any two points in X are connected by a path in $\mathcal{L}_X$. Since X is connected, it suffices to verify the claim locally. Let U is G_x-invariant neighborhood of x satisfying (29), such that $V = p(U)$ is an open metric ball in Y centered at $y = p(x)$. Take $u \in U$, $v := p(u) \in V$. Let $c : [0,T] \to V$ be a geodesic connecting v to y. Then there exists a lift $\tilde{c} : [0,T] \to U$ of c with $\tilde{c}(0) = u$. Since $x \in U$ is the only point projecting to y, we get $\tilde{c}(T) = x$. By taking concatenations of pairs of such radial paths in U, we conclude that any two points in U are connected by a path $\tilde{c} \in \mathcal{L}_X$.

Given a length structure on X, one defines a path-metric (metrizing the topology of X) by

$$d_X(x_1, x_2) = \inf_{\gamma}\{\ell(\gamma)\}$$

where the infimum is taken over all $\gamma \in \mathcal{L}_X$ connecting x_1 to x_2. By the construction, the projection $p : (X, d_X) \to (Y, d_Y)$ is 1-Lipschitz.

Lemma 30. *The metric d_X is complete.*

Proof. Let (x_n) be a Cauchy sequence in (X, d_X). By the construction of the metric d_X, there exists a finite length path $\tilde{c} : [0, 1) \to (X, d_X)$ and a sequence $t_n \in [0, 1)$ such that $\tilde{c}(t_n) = x_n, \tilde{c}(0) = x = x_1$. Since the map p is 1-Lipschitz, the path $c = p \circ \tilde{c} : [0, 1) \to (Y, d_Y)$ also has finite length. Since the metric d_Y was complete to begin with, the path c extends to a path $\bar{c} : [0, 1] \to Y$; set $y' := \bar{c}(1)$.

Assume for a moment that G acts freely on X. Then we have the *uniqueness* of lifts of paths from Y to X. Thus, the unique lift $\tilde{\bar{c}}$ of $\bar{c}$ starting at the point x satisfies the property that its restriction to $[0, 1)$ equals $\tilde{c}$. It follows that the sequence (x_n) converges to $\tilde{\bar{c}}(1)$. Below we generalize this argument to the case of non-free actions.

Let U be a neighborhood of $y' = \bar{c}(1)$ which is the projection to Y of a relatively compact slice neighborhood $\tilde{U}$ of some $x' \in p^{-1}(y')$. Without loss of generality (by removing finitely many initial terms of the sequence (x_n)) we can assume that the image of the path c lies entirely in U. Applying the path-lifting property to the path c with the prescribed *terminal* point x', we obtain a lift of the path $\bar{c}$ that terminates at x'. This lift has to be entirely contained in $\tilde{U}$ and its initial point has to be of the form $g(x)$ for some $g \in G$. Applying g^{-1} to this lift, we obtain another lift of $\bar{c}$, denoted $\tilde{\bar{c}}$, which starts at x and terminates at $g^{-1}(x')$.

Consider the restriction of $\tilde{\bar{c}}$ to $[0, 1)$. This restriction is also a lift to the path $c|_{[0,1)}$ and the image of the latter lies entirely in U. Hence, the image of $\tilde{\bar{c}}|_{[0,1)}$ lies entirely in the relatively compact subset $g^{-1}(\tilde{U}) \subset X$. Thus, the Cauchy sequence (x_n) lies in a relatively compact subset of X, and it follows that this sequence converges in X. □

Since (X, d_X) is locally compact and complete, by Theorem 2.5.28 (and Remark 2.5.29) in [8], (X, d_X) is a geodesic metric space. Lastly, we note that, by the construction, the length structure on X and, hence, the metric d_X, is G-invariant. This concludes the proof of the theorem. □

Question 31. *Local compactness and local connectivity were critical for the proof of the theorem. Does the theorem hold without these assumptions?*

7. Fundamental domains of properly discontinuous group actions

7.1. **Fundamental sets.** As with many notions going back to the 19th century, there is no consistency in the literature regarding the definition of fundamental sets and domains. The next definition follows [17]. Our definition is similar to the definition given by Borel and Ji in [5, Definition III.2.14], except that their local finiteness condition is weaker: It is required only for singletons K.

Definition 32. *A closed subset $F \subset X$ is a* fundamental set *for a proper action of a discrete G on a topological space X if $G \cdot F = X$ and for every compact $K \subset X$, the transporter set $(F|K)_G$ is finite (the* local finiteness condition*). A closed subset $F \subset X$ is a* fundamental set in the sense of Koszul *if, moreover, there exists an open neighborhood U of F such that for every compact $K \subset X$, the transporter set $(U|K)_G$ is finite.*

Fundamental sets appear naturally in the reduction theory of arithmetic groups (Siegel sets), see [24] and [5]. We note, however, that in the literature there are many alternative notions of fundamental sets, inconsistent with the one given above, see e.g. Beardon's book [3, 9.1]: According to Beardon's definition, a subset F of X is called fundamental for the action of G on X if F intersects every G-orbit in X in exactly one point. We will avoid using this definition since its set-theoretic nature provides us with no useful control of the structure of F.

The local finiteness condition in the definition of a fundamental set has several implications:

Lemma 33. *Suppose that $F \subset X$ is a fundamental set for a proper action of a discrete group G on a 1st countable and Hausdorff space X. Then:*

1. For every $x \in X$ there exists a neighborhood W of x such that $(F|U)_G$ is finite.

2. For every $x \in X$ there exist a finite subset $E = \{g_1, ..., g_k\} \subset G$ such that the interior of $g_1(F) \cup ... \cup g_n(F)$ is a neighborhood of x in X.

Proof. 1. Suppose that such W does not exist. Then there exists a sequence of distinct elements $g_n \in G$ and points $x_n \in X$ such that

$$\lim_{n\to\infty} x_n = x$$

and $x_n \in g_n(F)$. It follows that for the compact $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ the transporter set $(F|K)_G$ is infinite, which is a contradiction.

2. By the local finiteness condition, there are only finitely many elements $g_1, ..., g_k \in G$ such that $x \in g_i(F)$. By Part 1 of the lemma, there exists a neighborhood W of x such that $W \cap gF \neq \emptyset$ only for $g \in E = \{g_1, ..., g_k\}$. But then, since $GF = X$, it follows that

$$W \subset g_1F \cup ... \cup g_kF. \quad \square$$

For each fundamental set F of a G-action on a topological space X we define its quotient space F/G as the quotient space of the equivalence relation $x \sim y \iff Gx = Gy$. The following proposition explains why fundamental sets are useful: They allow one to describe quotient spaces of proper actions by discrete groups using less information than is contained in the description of the action.

Proposition 34. *Suppose that F is a fundamental set for a proper action by discrete group G on a 1st countable and Hausdorff space X. Then the natural projection map $p : F/G \to X/G$ is a homeomorphism.*

Proof. The map p is continuous by the definition of the quotient topology. It is also obviously a bijection. It remains to show that p is a closed map. Since F is closed, it suffices to show that the projection $q : F \to X/G$ is a closed map. Suppose that (x_n) is a sequence in F such that $q(x_n)$ converges to some $y \in X/G$, y is represented by a point $x \in F$. Then there is a sequence $h_n \in G$ such that $z_n = h_n(x_n)$ converges to x. If the sequence (h_n) contains infinitely many distinct elements, we obtain a contradiction with the local finiteness property of F similarly to the proof of Lemma 33. Hence, the set $E = \{h_n : n \in \mathbb{N}\}$ is finite. Applying inverses of the elements $h \in E$, to the sequence (z_n), we see that the subset $\{x_n : n \in \mathbb{N}\} \subset X$ is relatively compact. Thus, $q : F \to F/G$ is a closed map. $\square$

There are several existence theorems for fundamental sets. The next proposition, proven in [17, Lemma 2], guarantees existence of fundamental sets under the *paracompactness assumption* on X/G.

Proposition 35. *Each proper action $G \times X \to X$ of a discrete group G on a locally compact Hausdorff space X with paracompact quotient X/G admits a fundamental set in the sense of Koszul.*

Another construction of fundamental sets is given by *closed Dirichlet domains*. Let $G \times X \to X$ be an isometric proper action of a discrete group G on a metric space (X, d). The *closed Dirichlet domain* for this action is

$$\hat{D}_x = \{y \in X : d(y, x) \leq d(y, gx) \quad \forall g \in G\}. \tag{36}$$

Note that $g\hat{D}_x = \hat{D}_{gx}$. We also note that $\hat{D}_x$ is a closed subset of X since it is the intersection of a family of closed subsets

$$\{y \in X : d(y,x) \le d(y,gx)\}, g \in G.$$

Proposition 37. *Suppose that $G \times X \to X$ is a metrically proper isometric action of a discrete group G. Then every closed Dirichlet domain $\hat{D} = \hat{D}_x$ is a fundamental set for the G-action.*

Proof. 1. Let us prove that $g\hat{D} = X$. For each $y \in X$ the function $g \mapsto d(y,gx)$ is a proper function on G, hence, it attains its minimum at some $g \in G$. Then, clearly, $y \in \hat{D}_{gx} = g\hat{D}_x$. Thus, $g\hat{D} = X$.

2. Secondly, we verify local finiteness. Consider a metric ball $B = B(x,R)$ for any $R > 0$. If $\hat{D}_{gx} \cap B \ne \emptyset$, for every point y in this intersection

$$d(g^{-1}y, x) = d(y, gx) \le d(y,x) < R,$$

In view of metric properness of the G-action, the set of such elements $g \in G$ is finite. □

We will discuss Dirichlet domains (and their generalizations via Voronoi tessellations) again in Section 7.2.

Note that in the definition of a closed Dirichlet domain one does not really need a metric, what is needed is a G-invariant continuous function $d : X \times X \to \mathbb{R}_+$. For the proof of Proposition 37 to go through one needs a metric δ on X such that:

(a) The G-action is metrically proper on (X, δ).
(b) $\delta(y,x) \le \phi(d(y,x))$ for some function ϕ.

An example of the situation when this is useful appears in the context of discrete subgroups Γ of $G = SL(n,\mathbb{R})$ acting on the space X of symmetric positive-definite $n \times n$ matrices M with $\det M = 1$ by

$$M \mapsto g^T M g, g \in SL(n,\mathbb{R}).$$

Then Selberg in [23] used the function $d : X \times X \to \mathbb{R}_+$,

$$d(A,B) = \log\left(\frac{1}{n} tr(A^{-1}B)\right)$$

to define an analogue of Dirichlet domains for the Γ-action on X. (See also [15].) The advantage of such generalized Dirichlet domains is that they are intersections of X with polyhedral cones in the space of all symmetric $n \times n$ matrices.

Definition 38. *Suppose that $G \times X \to X$ is a continuous action. A closed subset $F \subset X$ is a* strict fundamental set *for the action if it intersects each G-orbit in X in exactly one point.*

Strict fundamental sets do not exist often, but they do exist for some classes simplicial group actions on simplicial complexes (one does not even need to assume properness), e.g. for actions of Coxeter groups on *Coxeter complexes* and actions of semisimple Lie groups (as well as semisimple algebraic groups over discrete valued fields) on *buildings* (see e.g. [22]). In the next section we will use a construction of strict fundamental sets for properly discontinuous simplicial group actions on vertex sets of connected graphs described below. Suppose that Γ is a simplicial graph (a 1-dimensional simplicial complex), $G \times \Gamma \to \Gamma$ is a simplicial action of a discrete group G. (The action need not be proper.) Note that the edge-stabilizers need not fix the invariant edges. However, if Γ' denotes the barycentric subdivision of Γ then the induced action of G on Γ' is *without inversions*, i.e. if an element of G preserves an edge, then it fixes the edge pointwise.

Lemma 39. *Suppose that Γ is connected. Then there exists a subtree $\Phi \subset \Gamma'$ such that the vertex set of Φ is a strict fundamental set for the G-action on the vertex set of Γ'.*

Proof. The quotient Γ'/G has natural structure of a connected simplicial graph. Let $q : \Gamma' \to \Gamma'/G$ denote the quotient map. Choose $T \subset \Gamma'/G$, a maximal subtree (this may require the Axiom of Choice if the vertex set of Γ'/G is uncountable). We will construct Φ by lifting T (inductively) to Γ'. We pick a vertex $v \in \Gamma'/G$ and lift it arbitrarily to a vertex $\tilde{v} \in q^{-1}(v) \in \Gamma'$. Then, of course, $G\{\tilde{v}\} \cap \{\tilde{v}\} = \{\tilde{v}\}$. We proceed inductively, working with subtrees $B_n \subset T$ which are closed metric balls of radius n centered at v. Suppose that we defined a subtree $\Phi_n \subset \Gamma'$ such that $q(\Phi_n) = B_n$ and each G-orbit in Γ' intersects Φ_n in at most one point. Let $e = [u, w]$ be an edge in B_{n+1} with $u \in B_n$. Then there exists an edge $\tilde{e} = [\tilde{u}, \tilde{w}]$ of Γ' which projects to e and $\tilde{u} \in B_n$ is a vertex projecting to u. We add the edge $\tilde{e}$ (and the vertex $\tilde{w}$) to Φ_n (note that $\tilde{w}$ cannot belong to Φ_n). We repeat this for all edges of B_{n+1} which are not in B_n, resulting in a subtree $\Phi_{n+1} \subset \Gamma'$. By the construction, each G-orbit in Γ' intersects Φ_{n+1} in at most one point. Lastly, the union

$$\Phi = \bigcup_n \Phi_n$$

is a subtree satisfying the required properties. ☐

Note that, unless $\Gamma'/G = T$ (i.e. Γ'/G is a tree), Φ is not a fundamental set of the G-action on Γ' since preimages of edges of Γ'/G that are not in T are not contained in the G-orbit of Φ.

7.2. **Fundamental regions and domains.** One frequently encounters a sharper version of fundamental sets, called *fundamental domains* or *fundamental regions*. Again, there is no consistency in this definition in the literature. Below is a small sample of existing definitions. Ratcliffe in [21, §6.6] defines *fundamental regions* for a properly discontinuous isometric G-action on a metric space (X, d) as *open* subsets $R \subset X$ such that $X = G\overline{R} = X$ and $gR \cap R = \emptyset$ for all $g \in G \setminus \{1\}$. Then Ratcliffe defines *fundamental domains* as connected fundamental regions. Ratcliffe also defines *locally finite* fundamental domains by imposing the extra assumption of local finiteness just as in Definition 32 given above. Beardon in [3, §9.1, 9.2] also defines fundamental domains as open connected subsets as above, but (working in the context of subsets of hyperbolic spaces) imposes the extra condition that the boundary has Lebesgue measure zero. In contrast, S. Katok in §3.1 of [16], defines fundamental regions $F \subset X$ as closures of certain open subsets $R \subset X$ where R is a fundamental region as in Ratcliffe's definition. Furthermore, Benedetti and Petronio, [4, §C1], define fundamental domains as *Borel subsets* $F \subset X$ such that $GF = X$ and $gF \cap F \subset \partial F$ for all $g \in G \setminus \{1\}$.

Below we will adopt a variation of Ratcliffe's and Katok's terminology of *fundamental regions/domains* but impose the local finiteness condition from the beginning.

Definition 40. *1. A subset U of a topological space is called an* open domain *(or a* regular open subset*) if U is the interior of its closure.*

2. A subset V of a topological space is called a closed domain *(or a* regular closed subset*) if V is the closure of its interior.*

Definition 41. *Suppose that $G \times X \to X$ is a proper action of a discrete group on a topological space X.*

1. An open subset $R \subset X$ is an open fundamental region *for this action if the following hold:*

(1) $G \cdot \overline{R} = X$.

(2) $gR \cap R \neq \emptyset$ *if and only if* $g = 1$.

(3) *For every compact subset $K \subset X$, the transporter set $(\overline{R}|K)_G$ is finite, i.e. the family $\{g\overline{R}\}_{g\in G}$ of subsets in X is* locally finite.

2. A closed subset $F \subset X$ is a closed fundamental domain *if F is a closed domain in X and*

(1) $G \cdot F = X$.

(2) $g\,\mathrm{int}(F) \cap \mathrm{int}(F) \neq \emptyset$ *if and only if* $g = 1$.

(3) *For every compact subset $K \subset X$, the transporter set $(F|K)_G$ is finite.*

Below we describe the most common construction of open fundamental regions, *open Dirichlet domains* and their variations.

Given an isometric metrically proper action of a discrete group G on a metric space (X, d) and a point $x \in X$, one defines the *open Dirichlet domain* of the action as

$$D_x = \{y \in X : d(y, x) < d(y, gx) \quad \forall g \in G \setminus G_x\}.$$

Thus, $D_x \subset \hat{D}_x$ (see (36)). It is clear from the construction that for every $g \in G \setminus G_x$,

$$gD_x \cap D_x = \emptyset$$

and $gD_x = D_x$ for all $g \in G_x$. In order to have a chance to get a fundamental region using open Dirichlet domains one has to assume that $G_x = \{1\}$.

Remark 42. *Suppose that G is countable, X is a complete metric space, $G\times X \to X$ is a continuous action and fixed point sets in X of nontrivial elements of G are nowhere dense. Then Baire's Theorem implies existence of $x \in X$ such that $G_x = \{1\}$. For instance, if X is a connected topological manifold, G is discrete and acts effectively and properly on X. Then the fixed-point set of each nontrivial element of G has empty interior, see* [19].

More generally, given a closed discrete subset $E \subset X$, one defines the *Voronoi tessellation* $\mathcal{V}_E$ of X corresponding to E. The open/closed tiles of the tessellation are the subsets $V_x, \hat{V}_x, x \in E$, of X defined as

$$V_x = \{y \in X : d(y, x) < d(y, x') \quad \forall x' \in E \setminus \{x\}\},$$
$$\hat{V}_x = \{y \in X : d(y, x) \leq d(y, x') \quad \forall x' \in E \setminus \{x\}\}.$$

The point x is the *center* of the tiles $V_x, \hat{V}_x$. Each $\hat{V}_x$ is closed in X (as the intersection of closed subsets). The *open tile* V_x need not be an open subset of X (the intersection of open subsets need not be open). A sufficient condition is that $E \subset X$ is *metrically proper*, see Section 6.

Lemma 43. *If $E \subset X$ is metrically proper then each open tile V_x of $\mathcal{V}_E$ is an open subset of X and the collection of tiles $\hat{V}_x, x \in E$, is locally finite.*

Proof. 1. Take $y \in V_x$. In view of metric properness of E, the function

$$d(y, \cdot) - d(y, x) : E \setminus \{x\} \to \mathbb{R}_+$$

attains its positive minimum R at some $x' \in E \setminus \{x\}$. Then $B(y, R/2) \subset V_x$.

2. Consider a unit ball $B = B(z, 1) \subset X$. Suppose that $\hat{V}_x \cap B \neq \emptyset$ for some $x \in E$. Then, whenever $\hat{V}_y \cap B \neq \emptyset, y \in E$, $d(y, z) \leq d(x, z) + 1$. By the metric properness of E, the number of such points $y \in E$ is finite. □

The key issue that we will have to deal with is that, even if E is metrically proper, the closed tile $\hat{V}_x$ is not necessarily the closure of the open tile V_x. Moreover, in general, the bisectors

$$Bis(x, z) = \{y \in X : d(y, x) = d(y, z)\}$$

may have nonempty interior in X. This happens, for instance, in the case of metric graphs.

Example 44. *Consider the space X which is the union of two coordinate lines in $\mathbb{R}^2$, with the induced path-metric d, i.e. the restriction of the ℓ_1-metric from $\mathbb{R}^2$. Thus, (X, d) is a complete geodesic metric space. Let $G = \mathbb{Z}_2$, whose generator g acts on X by restriction of the antipodal map $(x, y) \mapsto (-x, -y)$ on $\mathbb{R}^2$. The group G has unique fixed point in X, namely the origin $\mathbf{0} = (0, 0)$. For every point $p \in X \setminus \{\mathbf{0}\}$ the closed Dirichlet domain $\hat{D}_p$ is the union of three coordinate rays, while D_p consists of just one open coordinate ray. In particular, $\delta D_p = \hat{D}_p \setminus D_p$ is a coordinate line and, thus, is not contained in the boundary of D_p (which is the singleton $\{\mathbf{0}\}$). The interior of $\hat{D}_p$ is $\hat{D}_p \setminus \{\mathbf{0}\}$, hence,*

$$\operatorname{int} \hat{D}_p \cap g(\operatorname{int} \hat{D}_p)$$

is nonempty and equals a coordinate line minus the origin. In particular, the interior of any closed Dirichlet domain cannot be a fundamental region. Note also that the closure of D_p is the closed coordinate ray containing p, which implies that $G\overline{D}_p \neq X$ (it misses two open coordinate rays). Of course, in this example one can take a suitable open subset of $\operatorname{int} \hat{D}_p$ (the union of two open rays) as a fundamental region. However, it cannot be chosen to be connected. Thus, connectedness of fundamental regions (as required by Ratcliffe's definition of a fundamental domain) is an unreasonable requirement in the setting of general complete geodesic metric spaces.

Below we discuss some basic properties of Voronoi tiles.

Lemma 45. *Let $\phi : X \to (0, \infty)$ be an L-Lipschitz function for some $L \leq 1/2$ and let $E \subset X$ be such that for every $x \in X$, the open ball $B(x, \phi(x))$ has nonempty intersection with E. Then for every $x \in X$, $\hat{V}_x \subset B(x, 2\phi(x))$.*

Proof. Take $y \in \hat{V}_x$. Then there exists $z \in E$ such that $d(y, z) < \phi(y)$. Since $y \in \hat{V}_x$, $d(x, y) \leq d(z, y) < \phi(y)$. By the L-Lipschitz property of ϕ, we have $\phi(y) \leq \phi(x) + Ld(x, y)$, implying

$$(1 - L)d(x, y) < \phi(x)$$

and $d(x, y) < \frac{1}{1-L}\phi(x) \leq 2\phi(x)$. Therefore, $y \in B(x, 2\phi(x))$. □

Suppose that (X, d) is a metric space, $G \times X \to X$ an isometric properly discontinuous action, $E \subset X$ is a metrically proper G-invariant subset, $S = q(E) \subset X/G$ is the image of E under the quotient map $q : X \to X/G$. We then have two Voronoi tessellations $\mathcal{V}_E$ (of X) and $\mathcal{V}_S$ (of X/G equipped with the metric d_G).

Lemma 46. *For every closed and every open Voronoi tile $\hat{V}_x, V_x, x \in E$, we have $q(\hat{V}_x) = \hat{V}_{[x]}$ and $q(V_x) = V_{[x]}$.*

Proof. The statement is a direct consequence of definitions of Voronoi tiles and the metric d_G. □

Our next goal is to find a condition on E that ensures injectivity of the restriction of q to each $\hat{V}_x$. Recall that in Section 6, given an isometric properly discontinuous action $G \times X \to X$, we defined a function $\rho : X/G \to \mathbb{R}_+$ (as well as $\rho : X \to \mathbb{R}_+$).

Lemma 47. *Suppose now that G is a group acting freely, isometrically and properly discontinuously on (X, d). Suppose that $E \subset X$ is a G-invariant closed discrete subset such that $B(x, \frac{1}{4}\rho(x)) \cap E \neq \emptyset$ for all $x \in X$. Then the quotient map $q : X \to X/G$ is injective on each closed tile $\hat{V}_x, x \in E$. In other words,*

$$\hat{V}_x \cap \hat{V}_{gx} = \emptyset, \ \forall x \in E, \ \forall g \in G \setminus \{1\}.$$

Proof. For every $y \in \hat{V}_x$ there exists $x' \in E$ at distance $< r = \frac{1}{4}\rho(y)$ from y. Since $y \in \hat{V}_x$, we have $d(x, y) < r$ and, therefore, $d(x, x') < 2r$. Using the inequality

$$|\rho(x) - \rho(y)| < 2r,$$

we get $d(x, gx) \geq \rho(x) > \rho(y) - 2r = 2r$ for all $g \in G \setminus \{1\}$. By the triangle inequality, $d(x, y) < r$ implies that $d(y, gx) > r$. Thus, $y \notin \hat{V}_{gx}$. $\square$

We will construct subsets $E \subset X$ satisfying the assumptions of Lemma 47 in Lemma 56.

A subset A of a geodesic metric space (X, d) is *starlike* with respect to a point $a \in A$ if for each $x \in A$ every geodesic segment ax is contained in A.

Lemma 48. *Suppose that (X, d) is a geodesic space and $\mathcal{V}_E$ be the Voronoi tessellation corresponding to a metrically proper subset $E \subset X$. Then each tile $V_x, \hat{V}_x$ of $\mathcal{V}_E$ is starlike with respect to its center.*

Proof. The proof is essentially the same as the one in [21, Theorem 6.6.13]. Take a point $z \in \hat{V}_x$ and let $c : [0, T] \to X$ be a geodesic connecting x to z. Then for each $t \in [0, T]$ and $y \in E \setminus \{x\}$, we get (by the triangle inequality)

$$d(x, c(t)) = t = T - d(c(t), z) = d(x, z) - d(c(t), z) \leq d(y, z) - d(c(t), z) \leq d(y, c(t)).$$

Hence $c(t) \in \hat{V}_x$ and, therefore, $\hat{V}_x$ is starlike with respect to x. The same argument works for V_x. $\square$

The basic examples of Voronoi tessellations are when (X, d) is a Euclidean or a real-hyperbolic space; in these cases Voronoi tiles (and, hence, their intersections) are convex. This need not be the case in general even when one works with, say, *complex-hyperbolic spaces* (see e.g. [12]). Below we will see that some kind of convexity still holds in the case of Voronoi tessellations of Gromov-hyperbolic spaces.

Recall that a subset Y of a geodesic metric space (X, d) is called λ-*quasiconvex* if every geodesic segment xy with the end-points in Y is contained in the closed λ-neighborhood of Y, i.e. $d(z, Y) \leq \lambda$ for all $z \in xy$.

Corollary 49. *Suppose, additionally, that (X, d) is δ-hyperbolic. Then:*

1. *For every Voronoi tessellation $\mathcal{V}_E$ of X, each tile $V_x, \hat{V}_x$ is δ-quasiconvex.*
2. *Each bisector $Bis(x, y)$ in X is 2δ-quasiconvex.*

Proof. 1. This is a direct consequence of Lemma 48 and the definition of δ-hyperbolicity via slimness of geodesic triangles.

2. Take $E = \{x, y\}$ and the corresponding Voronoi tessellation of X with just two closed tiles, $\hat{V}_x, \hat{V}_y$. Then $Bis(x, y) = \hat{V}_x \cap \hat{V}_y$. Suppose that points p, q belong to $Bis(x, y)$. Consider a point z on a geodesic pq in X. By Part 1, there exist points $x' \in \hat{V}_x, y' \in \hat{V}_y$ within distance δ from z. In particular, $d(x', y') \leq 2\delta$. Since the geodesic $x'y'$ connects $\hat{V}_x, \hat{V}_y$, by continuity of the function $d(x, \cdot) - d(y, \cdot)$, there exists $z' \in x'y' \in Bis(x, y)$. By the triangle inequality, $d(z, z') \leq 2\delta$. $\square$

Note that the proof of Lemma 48 also shows that for each $z \in \hat{V}_x$ and $t < T$, either we get the strict inequality $d(x, c(t)) < d(y, c(t))$, or $c(t)$ belongs to a geodesic yz. If the former case occurs for all $y \in E \setminus \{x\}$, we conclude that we get $c(t) \in V_x$. In particular, in that case, $\hat{V}_x$ is the closure of V_x. In order to rule out the second possibility ($c(t)$ belongs to a geodesic yz) one has to impose extra restrictions.

Definition 50. *A geodesic metric space (X, d) has* nonbranching geodesics *if each geodesic $c : I \to (X, d)$ is uniquely determined by its restriction to a nonempty open subset of the interval I.*

For instance, geodesics in Riemannian manifolds and, more generally, manifolds with smooth Finsler metrics and Alexandrov spaces satisfy this property.

Corollary 51. *Suppose that (X, d) is a metric space with nonbranching geodesics. Then for each Voronoi tessellation $\mathcal{V}_E$ of X and every $x \in E, z \in \hat{V}_x$ and geodesic $c : [0, T] \to xz$ connecting x to z, one has $c(t) \in V_x$, $t < T$. In particular, $\hat{V}_x$ is the closure of V_x. Moreover, V_x is an open domain in X.*

Proof. The proof is the same as the one in [21, Theorem 6.6.13]. Suppose that $c(t) \notin V_x$ for certain $t < T$. Then for all $s \in [t, T]$ we have $c(s) \in \delta V_x = \hat{V}_x \setminus V_x$. Due to the local finiteness of $\mathcal{V}_E$, there exist $y \in E \setminus \{x\}$ and t', $t < t' < T$, such that $c(s) \in \hat{V}_y$ for all $s \in [t', T] \subset [t, T]$. Therefore, for all $s \in [t', T]$ we get

$$s = d(x, c(s)) = d(y, c(s)),$$

and, thus, $d(y, c(s)) + d(c(s), z) = d(y, z)$. In other words, the concatenation of geodesics $yc(t') \star c(t')z$ is a geodesic γ in (X, d). For the second segment $c(t')z$ of this geodesic we will take the restriction of c to $[t', T]$. Since $y \neq x$, geodesics c and γ have distinct images; on the other hand, they agree on the open subinterval (t', T). This contradicts the nonbranching assumption. The proof that V_x is an open domain in X is similar and we omit it. □

Corollary 52 (See Theorem 6.6.13 in [21])**.** *Suppose that (X, d) is a metric space with nonbranching geodesics, $G \times X \to X$ is an isometric metrically proper action. Then for every $x \in X$ with $G_x = \{1\}$ the open Dirichlet domain D_x is a connected open fundamental region for the G-action on X. Moreover, $\hat{D}_x$ is the closure of D_x and the interior of $\hat{D}_x$ is precisely D_x.*

Proof. We apply Corollary 51 to the Voronoi tessellation $\mathcal{V}_{Gx}$. Corollary 51 implies that $\hat{D}_x$ is the closure of D_x. Connectedness of D_x is clear from the same corollary. The fact that $G\hat{D}_x = X$ follows from Proposition 37. It remains to prove that the interior of $\hat{D}_x$ is D_x. Take $z \in \hat{D}_x \cap \hat{D}_y$, where $y \in Gx \setminus \{x\}$. By applying Corollary 51 to the Voronoi tile $\hat{D}_y$ we see that z belongs to the closure of D_y. Since the latter is disjoint from $\hat{D}_x$, we conclude that $z \notin \operatorname{int} \hat{D}_x$. □

Question 53. *Suppose that M is a connected topological manifold. Does M admit a complete geodesic metric with nonbranching geodesics?*

In the rest of the section we will prove existence of connected closed fundamental domains for free properly discontinuous actions on geodesic metric spaces by using Voronoi tessellations more general than the ones given by the Dirichlet construction. (More precisely, we will use $\mathcal{V}_E$ for some G-invariant closed discrete subset E of X.) In what follows, (X, d) is a separable geodesic complete metric space.

Lemma 54. *Fix a point $y \in X$. Then there is a G_δ-subset $X_y \subset X$ consisting of points x such that $Bis(x, y)$ has empty interior.*

Proof. First of all, we prove that for every $z \in X$ the subset $\{x \in X : z \notin Bis(x, y)\}$ is open and dense in X. Openness is clear. To prove denseness, take a sequence of points $x_n \in xz \setminus \{x\}$ converging to x and note that for every x_n, $z \notin Bis(x_n, y)$.

Now, we take a dense countable subset $Z \subset X$ (here we use the separability assumption). Define

$$X_y = \{x : \forall z \in Z, z \notin Bis(x, y)\}.$$

Then X_y is the intersection of a countable family of open and dense subsets, i.e. is a G_δ-subset of X. Let us prove that for every $x \in X_y$ the bisector $Bis(x,y)$ has empty interior. Take $w \in Bis(x,y)$ and a sequence $z_n \in Z$ converging to w. Then, by the definition of the set X_y, for all n, $z_n \notin Bis(x,y)$. Hence, w cannot belong to the interior of $Bis(x,y)$. □

Since the intersection of a countable family of G_δ-subsets is again a G_δ-subset, we obtain:

Corollary 55. *Let $Y \subset X$ be a countable subset. Define $X_Y = \bigcap_{y\in Y} X_y$. Then X_Y is a G_δ-subset of X. For every $x \in X_Y$ and $y \in Y$ the bisector $Bis(x,y)$ has empty interior.*

Lemma 56. *Suppose that (M,d) is a separable metric space, $\phi : M \to (0,\infty)$ a continuous function. Then there exists a countable discrete and closed subset $C \subset M$ such that for all $z \in M$, $B(z,\phi(z)) \cap C$ is nonempty.*

Proof. Take a maximal subset S of M satisfying the property that for all distinct $x, y \in S$,

$$d(x,y) \geq \frac{1}{2}\min(\phi(x), \phi(y)).$$

Separability of M ensures that such a subset is countable. Suppose that (x_n) is a sequence of distinct elements in S converging to $x \in M$. By continuity of ϕ, we have that

$$\phi(x_n) > \epsilon = \frac{1}{2}\phi(x) > 0$$

for all sufficiently large n. We have (for all $m \neq n$)

$$d(x_m, x_n) \geq \frac{1}{2}\min(\phi(x), \phi(y)) > \epsilon.$$

But then the sequence (x_n) cannot converge.

Lastly, take $x \in M$. Suppose that $B(x,\phi(x)) \cap S = \emptyset$. Then $d(x,y) \geq \phi(x)$ for all $y \in S$, which implies

$$d(x,y) \geq \min(\phi(x), \phi(y)).$$

Hence, $S \cup \{x\}$ still satisfies the inequality defining S. This contradicts the maximality of S. □

Addendum 57. *Assume, additionally, that (M,d) is a proper metric space. Then the collection of Voronoi tiles $\hat{V}_x, x \in C$, associated with the subset C in the lemma, is locally finite.*

Proof. This follows from the fact that each metric ball $B(x,R)$ in M contains only finitely many points from C, see Lemma 43. □

We next perturb the subset C in Lemma 56 to a new subset C' which satisfies essentially the same properties as C but also has empty interior of $\delta V_{x'} := \hat{V}_{x'} \setminus V_{x'}$ for every $x' \in C'$:

Lemma 58. *Suppose that (M,d) is a complete separable geodesic metric space. Then there exists a countable discrete and closed subset $C' \subset M$ such that for all $z \in M$, $B(z, 2\phi(z)) \cap C$ is nonempty and for each $x' \in C'$, $\delta V_{x'}$ has empty interior.*

Proof. We will take the subset C constructed in the previous lemma and perturb it inductively using Corollary 55 so that the bisectors $Bis(x',y') \subset M$ (for distinct points $x', y' \in C'$) have empty interior. Since the complements $\delta V_{x'}$ are contained in the union of bisectors $Bis(x',y'), y' \in C' \setminus \{x'\}$, it will follows that each $\delta V_{x'}$ has empty interior.

In order to construct the perturbation, using continuity and nonvanishing of ϕ, for every $x \in C$ we find $\epsilon_x >$ such that:

1. For all $z \in M$ satisfying $x \in B(z,\phi(z))$ we have $\epsilon_x < \phi(z)$.

2. $\lim_{n\to\infty} \epsilon_{x_n} = 0$ for some enumeration of the set C.
3. The metric balls $B(x, \epsilon_x)$, $x \in C$, are pairwise disjoint.

Now, we replace each $x \in C$ with some $x' \in B(x, \epsilon_x)$; the resulting subset C' of M is still countable. Furthermore, for every $z \in M$, there exists $x' \in C'$ as above such that $d(x', z) < 2\phi(z)$. If C' fails to be closed and discrete, there exists a sequence x'_n of distinct elements of C' converging to some $x \in M$. By (2),

$$\lim_{n\to\infty} d(x_n, x'_n) = 0, \quad x_n \in C.$$

Then the sequence (x_n) also converges to x and, by (3), all the points x_n are distinct. This is a contradiction. □

Lemma 59. *Suppose, additionally, that (M, d) is a proper metric space (separability is automatic in this case). Then for every point $y \in M$ there exists $x \in C'$ such that $y \in \overline{V}_x$.*

Proof. By the local finiteness of the Voronoi tessellation, for each point $y \in M$ there is a neighborhood U of y which intersects only finitely many closed tiles $\hat{V}_{x_i}, i = 1, ..., n$. Suppose that

$$y \in \delta V_{x_i}, i = 1, ..., n.$$

Then, by shrinking the neighborhood U further, we can assume that $U \subset \delta V_{x_1} \cup ... \cup \delta V_{x_n}$. But this contradicts the fact that each δV_{x_i} has empty interior. □

We now consider the situation when (X, d) is a proper geodesic metric space, $G \times X \to X$ is a free, isometric, properly discontinuous action. We form the quotient space $(M, d_M) = (X/G, d_G)$. This space is again proper and geodesic, see Lemma 21. Take the function $\phi = \frac{1}{16}\rho : M \to \mathbb{R}_+$, where ρ is defined via the G-action on X as in Section 6. Using Lemma 58 we find a suitable (countable) metrically proper subset $C' \subset M$. Let $E \subset X$ denote $q^{-1}(C')$, where $q : X \to X/G = M$ is the quotient map. We obtain two Voronoi tessellations, $\mathcal{V}_E, \mathcal{V}_{C'}$ of X and M respectively. According to Lemmata 46 and 47, the map q sends each closed tile $\hat{V}_x, x \in E$, of $\mathcal{V}_E$ homeomorphically to the closed tile $\hat{V}_{[x]}$ of $\mathcal{V}_{C'}$. By applying Lemma 59, we obtain:

Corollary 60. *For every point $y \in X$ there exists $x \in E$ such that $y \in \overline{V}_x$.*

We are now ready to prove the main theorem of this section:

Theorem 61. *Suppose that (X, d) is a proper geodesic metric space, $G \times X \to X$ is a free, isometric, properly discontinuous action. Then this action admits an open fundamental region R and a closed connected fundamental domain F.*

Proof. We define a G-invariant subset $E \subset X$ as above and pick its subset $S \subset E$ intersecting each G-orbit in E in exactly one point. Take

$$R = \bigcup_{x \in S} V_x.$$

We claim that R is an open fundamental region for the G-action on X. First of all, since G preserves the open and closed tiles of the Voronoi tessellation $\mathcal{V}_E$, we have that for every $x \in S$, $gV_x \cap V_x \neq \emptyset$ if and only if $gx = x$, i.e. $g = 1$ (since g acts freely on X). Next, the tiling $\mathcal{V}_E$ is locally finite (since the subset E is properly embedded in X). Hence, the collection of closures $\overline{V}_x, x \in E$, is locally finite as well. Lastly, by Corollary 60, each point $y \in X$ belongs to some $\overline{V}_x, x \in E$. Then there exists $z \in S$ such that $g(z) = x$ and, hence, $g\overline{V}_z = \overline{V}_x$. Thus, all properties of an open fundamental region are satisfied by R.

We construct a closed connected fundamental domain F by modifying the construction of R above. The main issue is that for a random choice of S, the region R need not have connected closure. We will use Lemma 39 to choose S more carefully. We let Γ denote the *incidence graph* of the closed cover $\{\overline{V}_x : x \in E\}$ of X: Vertices of Γ are the elements of E and we connect distinct points $x, y \in E$ by an edge if and only if $\overline{V}_x \cap \overline{V}_y \neq \emptyset$. Since X is connected, the graph Γ is connected as well. By the construction, the graph Γ is simplicial and the group G acts on Γ by simplicial automorphisms. We let Γ' denote the barycentric subdivision of Γ and $\Phi \subset \Gamma'$ a subtree given by Lemma 39. We let $S \subset E$ denote the intersection of E with the vertex set of Φ (i.e. the subset of vertices of Φ which are vertices of Γ). According to Lemma 39, each G-orbit in E intersects S in exactly one point. Thus, the open subset $R \subset X$ defined as above using S is an open fundamental region for the G-action on X. We let

$$F := \bigcup_{x \in S} \overline{V}_x.$$

Let us verify connectedness of F. First of all, each $\overline{V}_x$ is connected since V_x is connected. For any two points $x, y \in S$ there exists a vertex-path $x_1 x_2 ... x_n$ in Γ connecting these vertices ($x_1 = x, x_n = y$) such that each vertex of this path is in S (this follows from the fact that the graph Φ is connected). The union

$$\overline{V}_{x_1} \cup ... \cup \overline{V}_{x_n}$$

is connected since each intersection $\overline{V}_{x_i} \cap \overline{V}_{x_{i+1}}$ is nonempty. Thus, F is connected. The fact that F is closed follows from the fact that each $\overline{V}_x$ is closed and the family $\{\overline{V}_x : x \in S\}$ is locally finite in X. The subset

$$R = \bigcup_{x \in S} V_x$$

is open in X and dense in F. Hence, F is a closed domain in X. □

As an application we will prove existence of open fundamental regions and closed connected fundamental domains for free properly discontinuous group actions on a certain class of topological spaces, cf. [25].

Theorem 62. *Suppose that X is a 2nd countable, connected and locally connected locally compact Hausdorff topological space. Suppose that $G \times X \to X$ is a free proper action of a discrete countable group. Then this action admits an open fundamental region and a closed connected fundamental domain.*

Proof. In Theorem 25 we constructed a complete G-invariant geodesic metric d on X. The metric space (X, d) is necessarily proper, since X is locally compact. Using Theorem 61 we find an open fundamental region R and a closed connected fundamental domain F for the G-action on X. □

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