Winter 2008: MA Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose that G is a finitely-generated group and $n \in \mathbb{N}$. Show that G contains only finitely many subgroups of index $\leq n$.

Problem 2. Let A be an $n \times n$ complex matrix. Prove of disprove:

a. A is similar to its transpose.

b. If the sum of the elements of each column of A is 1, then 1 is an eigenvalue of A.

Problem 3. Recall that if R is a ring, an R-module M is projective means: If $f : A \to B$ is a homomorphism between two other R-modules, and if $g : M \to B$ is a homomorphism, then there is always a solution $h : M \to A$ to the equation $g = fh$. Prove that among $\mathbb{Z}$-modules, the only cyclic module $\mathbb{Z}/n$ which is projective is $\mathbb{Z}/0 = \mathbb{Z}$.

Problem 4. 2. Prove that $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices, has no non-trivial two-sided ideals.

Problem 5. Let A and B be two abelian groups with 25 elements. There is more than one possibility for A up to isomorphism, and likewise for B. Since all abelian groups are $\mathbb{Z}$-modules, we may tensor A and B as $\mathbb{Z}$-modules. What are the possibilities for the number of elements of $A \otimes B$?

Problem 6. Prove or disprove: The field $\mathbb{C}(x)$ of rational functions with complex coefficients, is a transcendental (i.e., non-algebraic) extension of the field $\mathbb{C}$.

Winter 2008: MA Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x).$$

(a) Show that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0,1]$.

(b) Show that $\sum_{n=0}^{\infty} |f_n|$ converges pointwise on $[0,1]$ but not uniformly.

Problem 2: Consider $X = \mathbb{R}^2$ equipped with the Euclidean metric,

$$e(x,y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} e(x,y) & \text{if } x,\ y \text{ lie on the same ray through the origin,} \\ e(x,0) + e(0,y) & \text{otherwise.} \end{cases}$$

Here, we say that x, y lie on the same ray through the origin if $x = \lambda y$ for some positive real number $\lambda > 0$.

(a) Prove that (X, d) is a metric space.

(b) Give an example of a set that is open in (X, d) but not open in (X, e).

Problem 3: Suppose that $\mathcal{M}$ is a (nonzero) closed linear subspace of a Hilbert space $\mathcal{H}$ and $\phi : \mathcal{M} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{M}$. Prove that there is a unique extension of ϕ to a bounded linear function on $\mathcal{H}$ with the same norm.

Problem 4: Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a (complex) Hilbert space $\mathcal{H}$ with spectrum $\sigma(A) \subset \mathbb{C}$ and resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. For $\mu \in \rho(A)$, let

$$R(\mu, A) = (\mu I - A)^{-1}$$

denote the resolvent operator of A.

(a) If $\mu \in \rho(A)$ and

$$|\nu - \mu| < \frac{1}{\|R(\mu, A)\|},$$

prove that $\nu \in \rho(A)$ and

$$R(\nu, A) = [I - (\mu - \nu)R(\mu, A)]^{-1} R(\mu, A).$$

(b) If $\mu \in \rho(A)$, prove that

$$\|R(\mu, A)\| \geq \frac{1}{d(\mu, \sigma(A))}$$

where

$$d(\mu, \sigma(A)) = \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is the distance of μ from the spectrum of A.

Problem 5: Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$.

Problem 6: Let $\Omega = \{x \in \mathbb{R}^3 \ : \ |x| < 1\}$ denote the unit ball in $\mathbb{R}^3$. Suppose that the sequences $\{f_k\}$ in $W^{1,4}(\Omega)$ and that $\{\vec{g}_k\}$ in $W^{1,4}(\Omega\,;\,\mathbb{R}^3)$. Suppose also that there exist functions $f \in W^{1,4}(\Omega)$ and $\vec{g}$ in $W^{1,4}(\Omega\,;\,\mathbb{R}^3)$, such that we have the weak convergence

$$f_k \rightharpoonup f \text{ in } W^{1,4}(\Omega)\,,$$
$$\vec{g}_k \rightharpoonup \vec{g} \text{ in } W^{1,4}(\Omega\,;\,\mathbb{R}^3)\,.$$

Show that there are subsequences $\{f_{k_j}\}$ and $\{\vec{g}_{k_j}\}$ such that we have the weak convergence

$$\vec{D}f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j} \rightharpoonup \vec{D}f \cdot \operatorname{curl} \vec{g} \quad \text{in} \quad H^{-1}(\Omega)\,.$$

Notation for Problem 6. Here f is a scalar function and $\vec{g} = (g_1, g_2, g_3)$ are three-dimensional vector-valued function. $\vec{D}$ denotes the three-dimensional gradient $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\operatorname{curl} \vec{g} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \times \vec{g}$

As customary, we use $H^{-1}(\Omega)$ to denote the dual space of the Hilbert space $H_0^1(\Omega)$ consisting of those functions in $H^1(\Omega)$ which vanish on the boundary (in the sense of trace). Two useful identities are that

$$\begin{aligned} \operatorname{curl}(\vec{D}f) &= 0 \quad \text{for any scalar function } f\,, \\ \operatorname{div}(\operatorname{curl} \vec{w}) &= 0 \quad \text{for any vector function } \vec{w}\,, \end{aligned}$$

where $\operatorname{div} \vec{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \partial_{x_3} F_3$ denotes the usual divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$.

Hint for Problem 6. Test $\vec{D}f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j}$ with a function $\psi \in H_0^1(\Omega)$ and use integration by parts to argue the weak convergence.