Fall 2001 Mathematics Graduate Program Masters Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Algebra

Problem 1. a. Prove Lagrange's theorem: If G is a finite group and H is a subgroup then $|H|$ divides $|G|$.
Prove or disprove: If n divides $|G|$ then there is a subgroup of G of order n.

Problem 2. a. Give an example of a group and a subgroup which is not normal.
b. Show that a group G of order 33 has a subgroup H of order 11.
c. Show that this subgroup H is normal.

Problem 3. a. Give an example of a noncommutative ring.
b. Let $Z[x]$ denote the ring of polynomials with integer coefficients. Prove or disprove: $Z[x]$ is a Principle Ideal Domain.

Problem 4. a. Give three examples of Field extensions of the rationals Q.
b. Let $Q(a)$ denote the field extension of the rationals obtained by adjoining a. Show that the field $Q\sqrt{2})$ is not isomorphic to $Q\sqrt{3})$.

Problem 5. a. Give an example of a finite field of order 3 and a finite field of order 9.
b. Give an example of an infinite field of characteristic 3.
c. Let F be a finite field. Show that the order of F is equal to p^n for some prime number p and positive integer n.

2. Analysis

Problem 6. Let $f : R \to R$ be a differentiable mapping with

$$\lim_{x \to \infty} f(x) = 0.$$

a. Show that there exists a sequence $x_n \to \infty$ with

$$\lim_{n \to \infty} f'(x_n) = 0.$$

b. Show that it is not necessarily true that $f'(x)$ is bounded.

Problem 7. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that $f(x) = x$ for some x.
Is the same true for a continuous function $f : (0, 1) \to (0, 1)$ on the open unit interval? Prove or give a counterexample.

Problem 8. Suppose $\lim_{n \to \infty} p_n = p_0$. Show that the set $E = \{p_0, p_1, p_2, \dots\}$ is compact.

Problem 9. Prove that $C[0, 1]$, the space of continuous functions on $[0, 1]$, is not complete in the L^1 metric: $\rho(f, g) = \int |f(x) - g(x)| dx$

Problem 10. Consider the map $T : R^3 \to R^3$ given by

$$T_1(x_1, x_2, x_3) = y_1 = x_1 \cos x_2$$
$$T_2(x_1, x_2, x_3) = y_2 = x_1 \sin x_2$$
$$T_3(x_1, x_2, x_3) = y_3 = x_3$$

a. Compute the Jacobian matrix of T.
b. For which values of $x = (x_1, x_2, x_3)$ is the map locally invertible (i.e. there exists a neighborhood U of x and a neighborhood V of $y = T(x)$ such that $T : U \to V$ is 1-1 and onto with inverse map $T^{-1} : V \to U$).
c. Compute the Jacobian matrix of T^{-1} at $f(x)$ where it exists.

3. Linear Algebra and other areas

Problem 11. a. Give an example of a real $n \times n$ matrix none of whose eigenvalues are real numbers.
b. Show that there is no such example which is 3×3.
c. Show that every eigenvalue of a symmetric real matrix is real.

Problem 12. a. Give three examples of linear mappings $L : R^3 \to R^3$ satisfying $L^2 = L$.
Let $P : R^n \to R^n$ be a linear mapping satisfying $P^2 = P$.
a. Show that every vector in R^n can be written as a sum of two vectors, one in the kernel of P and one in the image of P.
b. If P is symmetric (self-adjoint) then show that the image of P and the kernel of P are orthogonal subspaces of R^n.