PRELIMINARY EXAM IN ANALYSIS
SPRING 2020

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Suppose $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$. Show that $f \equiv 0$.

2. Suppose $f \in C^k(\mathbb{R}^n)$ where k is an integer $> n/2$. Suppose $(1+|\xi|^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^n)$ with $s > n/2$. Show that $\lim_{|x|\to\infty} f(x) = 0$.

3. Let $\mathcal{H}$ be a separable complex Hilbert space and $\{e_n\}_{n\geq 1}$ a orthonormal basis of $\mathcal{H}$. Define $\lambda_n = e^{i\pi/n}$, $n \geq 1$.

(a) Show that there is a unique $U \in \mathcal{B}(\mathcal{H})$ such that $Ue_n = \lambda_n e_n$.

(b) Show that U defined in a) is unitary.

(c) Show that the continuous spectrum of U is $\{1\}$.

4. Suppose $f \in L^2(\mathbb{R})$ and $\hat{f}$ is continuous. Suppose that $\hat{f}(\xi) = O(|\xi|^{-(1+\alpha)})$ as $|\xi| \to \infty$. Show that $|f(x+h) - f(x)| \leq C|h|^\alpha$ for all $h > 0$ and some constant C independent of h.

5. For each $n \in \mathbb{N}$, let

$$f_n(x) := \sum_{k=0}^{n-1} \chi_{[\frac{k}{n}, \frac{2k+1}{2n}]}(x).$$

Show that $f_n \rightharpoonup f$ weakly in $L^2([0,1])$ for some $f \in L^2([0,1])$. Is $f_n \to f$ (strongly) in $L^2([0,1])$?

6. Find a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ such that it is uniformly bounded and equi-continuous but fails to have a subsequence that converges uniformly on $\mathbb{R}$.