PRELIMINARY EXAM IN ANALYSIS, SPRING 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Consider the Hilbert space $L^2([0,2\pi])$ of complex-valued square integrable functions with the inner product given by

$$(f,g) = \int_0^{2\pi} \overline{f(x)}g(x)dx.$$

(a) For all $\phi \in \mathbb{R}$, define $g_\phi \in L^2([0,2\pi])$, by $g_\phi(\theta) = \sin(\theta - \phi)$, for $\theta \in [0,2\pi]$. Let V be the closed linear span of $\{g_\phi \mid \phi \in \mathbb{R}\}$. Show that V is two-dimensional.

(b) Find $k : [0,2\pi] \times [0,2\pi] \to \mathbb{C}$ such that for all $f \in L^2([0,2\pi])$, the integral operator K defined by

$$Kf(x) = \int_0^{2\pi} k(x,y)f(y)dy,$$

satisfies

$$\|Kf - f\| = \inf\{\|g - f\| \mid g \in V\}.$$

2. Let $S = [0,1] \times [0,1]$ and consider the space $C(S)$ of continuous complex-valued functions on S, equipped with the supremum norm. Define $F \subset C(S)$ by

$$F = \{f \mid \exists n \geq 1, g_1, \ldots, g_n, h_1, \ldots, h_n \in C([0,1]) \text{ such that } f(x,y) = \sum_{k=1}^{n} g_k(x)h_k(y)\}.$$

Show that F is dense in C(S).

3. Choose $f = a(2\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} - 1) \in L^2\mathbb{T} = L^2(-\pi,\pi]$ with a real and $f_n = f * f * \ldots * f$ the n-fold convolution so that $\{f_n\}$ converges in $\|\cdot\|_{L^2}$ to a nonzero function g. Find g.

4. Show that if K is a compact self adjoint linear operator on a separable Hilbert space with closed image then the image is finite dimensional.

5. Consider a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $|\nabla f| \in L^2(\mathbb{R}^2)$. Show that there exists a constant $K < 10$ such that the following inequality holds:

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leq K \|f\|_{L^1(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}.$$

6. Let $\Omega = (0,1) \subset \mathbb{R}$. For $\bar{u} = \int_\Omega u(x)\,dx$, show that

$$\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \|u'\|_{L^2(\Omega)}, \qquad \forall u \in W^{1,1}(\Omega).$$

(Hint: The average $\bar{u} = u(x_0)$ for some $x \in [0,1]$.)