PRELIMINARY EXAM IN ALGEBRA
SPRING 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. What is the splitting field of the polynomial $x^3 + x + 1 \in \mathbb{F}_5[x]$?

2. Prove that if G is a nontrivial nilpotent group, then its center $Z(G)$ is also nontrivial.

3. Let G be a group of order 50 and let n be the number of elements of order 5 in G. Find all possible values of n (and prove that the list is correct).

4. Consider the algebra $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Give a basis for A as a vector space over $\mathbb{R}$, and write out the product of every pair of basis vectors.

5. Give an example of a field F and a polynomial $f(x) \in F[x]$ which is irreducible but not separable. (Recall that a polynomial is *separable* if its has distinct roots in its splitting field.)

6. Calculate the character table (i.e., the table of the traces of complex irreducible representations) of the dihedral group D_4, by definition the finite group of order 8 with generators x, y and relations
$$x^4 = y^2 = 1 \qquad yxyx = 1.$$