Preliminary Exam in Analysis
Fall 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(n\pi x)dx = 0.$$

2. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\log x & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Be sure to justify your answer for each of the following questions.

(a) Is f Lipschitz continuous on $[0,1]$?

(b) Is f uniformly continuous on $[0,1]$?

(c) Suppose (p_n) is a sequence of polynomial functions on $[0,1]$, converging uniformly to f. Is the set $A = \{p_n \mid n \geq 1\} \cup \{f\}$ equicontinuous?

3. Show that for every $f \in C(\mathbb{T})$ and $\varepsilon > 0$ there is an initial condition $g \in C(\mathbb{T})$ for which there is a solution $u(x,t)$ to the heat equation on a ring with $u(x,0) = g(x)$ and $|u(x,1) - f(x)| < \varepsilon$ for every $x \in \mathbb{T}$.

4. Consider the functions $f_N(x) = (2\pi)^{-1}\sum_{n=-N}^{N} e^{inx}$. Show that if $g \in L^2(\mathbb{T})$ then $\{f_N * g\}$ converges in $\|\cdot\|_{L^2}$-norm to g (here, $*$ denotes convolution).

5. Show the following: For $u \in L^1(\mathbb{R}^n)$ there holds

$$\lim_{h\to 0} \|u(x+h) - u(x)\|_{L^1(\mathbb{R}^n)} = 0.$$

6. Let $\Omega = \{(x,y) : y \geq 0, x \in \mathbb{R}\}$. Let $f = C_c^1\left(\mathbb{R}^2\right)$ (space of continuous functions with compact support and with continuous first derivatives). Show the following

$$\int_{\mathbb{R}} |f(x,0)|^2 dx \leq 2\left(\int_{\Omega} |f(x,y)|^2 dxdy + \int_{\Omega} \left|\frac{\partial f}{\partial y}(x,y)\right|^2 dxdy\right).$$