PRELIMINARY EXAM IN ALGEBRA
FALL 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Prove that $\mathbb{Z}$ is a principal ideal domain.

2. Let $\mathbb{H}$ be the real quaternions. Then $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to which of the following rings? Prove your answer

(a) $\mathbb{C} \times \mathbb{C}$

(b) $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$

(c) $\mathbb{M}_2(\mathbb{C})$

(d) $\mathbb{M}_2(\mathbb{R})$

(e) $\mathbb{M}_2(\mathbb{H})$

(f) $\mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$

3. Let $f = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$. Compute the Galois group of f.

4. Solve the following:

(a) Prove that $R/I \otimes_R R/J \cong R/(I+J)$ for R a commutative ring, and $I, J \subset R$ are ideals.

(b) Find the dimension of $\mathbb{Q}[x,y]/(x^2+y^2) \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x,y]/(x+y^3)$ as a vector space over $\mathbb{Q}$, or explain why it is infinite.

5. Solve the following questions:

(a) If F is a field, prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.

(b) Show that $f(x) = x^2 + 2x + 2$ is irreducible in $\mathbb{Q}[x]$, and find the inverse of $1 + x$ in $\mathbb{Q}[x]/(f(x))$.

6. Let V be the subspace of $\mathbb{C}^3$ spanned by $v_1, v_2 = (1,-1,0), (0,1,-1)$, which is an invariant subspace under the permutation action of S_3, and so gives a two-dimensional representation $\rho : S_3 \to GL(V)$.

(a) Write down the matrices of $\rho(\sigma)$ in this basis.

(b) Describe a Hermitian inner product $\langle , \rangle$ on V in the basis v_1, v_2, which is G-invariant, i.e. $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$.

(c) Describe the tensor product $V \otimes_R V$, as an R-module where $R = \mathbb{C}[H]$ and $H \subset S_3$ is the subgroup $\{1, (123), (132)\}$.